Tutorials: Tue 10–12 Beckmann
Wed 14–16 Lin / Grosse-Beckmann

Affine varieties

k alg. closed field

Def. An aff alg subset in k^n is an $X \subseteq k^n$, common zero locus of a set of polynomials $M \subseteq k[x_1, \dots, x_n]$, i.e. $X = \{x \in k^n \mid \forall f \in M: f(x) = 0\} =: V(M)$

Rmk. a) $M \subseteq M' \Rightarrow V(M') \subseteq V(M)$

b) If I is the ideal gen'd by M then $V(I) = V(M)$.

Pf: $M \subseteq I \Rightarrow V(I) \subseteq V(M)$. Conversely, if $x \in V(M)$ and $f \in I$ then

$f = f_1 g_1 + \dots + f_m g_m$, $f_i \in M$, $g_i \in k[x_1, \dots, x_n]$

$\Rightarrow f(x) = f_1(x) g_1(x) + \dots + f_m(x) g_m(x) = 0$, $x \in V(I) \Rightarrow V(M) \subseteq V(I)$ □

c) For any set M, $V(M) = V(f_1, \dots, f_m)$ for some $f_1, \dots, f_m \in k[x_1, \dots, x_n]$

Pf: $V(M) = V(I)$ where $I = M\, k[x_1, \dots, x_n]$. By noetherianity, I fin. gen. (HBS) □

Prop./Def. The affine alg sets in k^n are the closed sets of a topology, called the Zariski topology. I.e., $U \subseteq k^n$ is open iff $k^n \setminus U = V(M)$ for some M.

Ex. $n = 1 \longrightarrow V(f) = \begin{cases} \text{finite set of pts} & \text{if } f \neq 0 \\ k & \text{if } f = 0 \end{cases}$

PF OF PROP: NTS closed sets are (i) closed under arbitrary intersection.

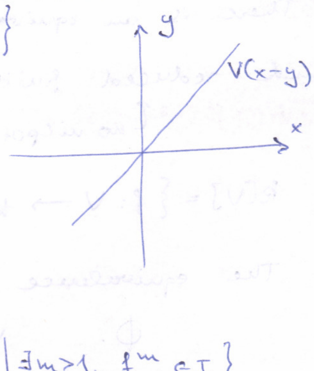
(ii) closed under finite union

(iii) $\emptyset$, k^n are closed.

This goes as it usually does. □

$$\left\{\begin{matrix}\text{ideals in the} \\ \text{polynomial ring}\end{matrix}\right\} \underset{I?}{\overset{V}{\rightleftarrows}} \{\text{aff. alg. sets}\}$$

We cannot go backwards: $V(x-y) = V((x-y)^2) = V((x-y)^3)$



Def. $I(X) := \{f \in k[x_1, \dots, x_n] \mid \forall x \in X: f(x) = 0\}$, X aff. alg. set.

E.g. $I(V(x-y)) = (x-y) = I(V((x-y)^k))$

Thm. (HNS) $\forall I \subseteq k[x_1, \dots, x_n]$: $I(V(I)) = \sqrt{I} := \{f \in k[x_1, \dots, x_n] \mid \exists m \geq 1, f^m \in I\}$

same **Cor.** $\{\text{aff alg sets in } k^n\} \xleftrightarrow{1:1} \{\text{radical ideals}\}$

Thm $\forall I \subseteq k[x_1, \dots, x_n]$: $I(V(I)) = \sqrt{I}$

Pf. in 2 steps.

Step 1. Weak Nullstellensatz. Every max ideal in $k[x_1, \dots, x_n]$ is of the form $(x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in k$ $\square$

Thm. (Noether Normalisation) R integral domain, f.g. over a field $k \Rightarrow$
$\Rightarrow \exists z_1, \dots, z_m \in R$ alg. indep. elements s.t. R is a fin. gen $k[z_1, \dots, z_n]$-module. $\square$

Step 2. Weak $\Rightarrow$ strong. $\square$

By the WNS, points correspond to max. ideals.

Def. A subset Y of a top space is irreducible if it cannot be written as a union of two proper subsets which are both closed in Y.

Ex. $V((x-y)\cdot(x+y)) = V(x-y) \cup V(x+y)$ not irred
k^n, $\emptyset$ irred.

Def. Affine variety: irreducible affine alg. set.

Lemma. $\mathfrak{a} \subset k[x_1, \dots, x_n]$ radical. Then $V(\mathfrak{a})$ irred. $\Leftrightarrow$ $\mathfrak{a}$ is prime. $\square$

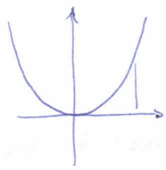
We thus get a correspondence $\underbrace{V = V_1 \cup \dots \cup V_e}_{\text{irreducibles}} \longleftrightarrow \underbrace{I = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_e}_{\text{radical primes}}$

Morphisms

Def. $V_1 \subseteq k^m$, $V_2 \subseteq k^n$ aff. alg. sets. Then $f : V_1 \longrightarrow V_2$ is a morphism if $\exists f_1, \dots, f_n \in k[x_1, \dots, x_n]$: $f(x) = (f_1(x), \dots, f_n(x)) \quad \forall x \in V_1$

Ex. $V(y - x^2)$

$$\left.\begin{array}{l} f(x,y) = x \\ f(x,y) = x + 5(y - x^2) \end{array}\right\} \longrightarrow \text{not unique!}$$



Def. V aff alg. subset. The coordinate ring of V is $k[V] := k[x_1, \dots, x_n] / I(V)$

Cor. There is an equivalence of ~~categories~~ contravariant categories b/w affine alg sets and their morphisms and reduced finitely generated k-algebras with k-algebra homomorphisms.
[no nilpotents]

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Rmk. $k[V] = \{ f : V \longrightarrow k^1 \text{ morphism} \}$

Rmk. The equivalence Φ in the above Cor. can be described as follows:

$$\Phi : \begin{array}{ccll} V & \longmapsto & k[V] & \text{on elements} \\ (f : V_1 \longrightarrow V_2) & \longmapsto & \left(\begin{array}{c} k[V_2] \longrightarrow k[V_1] \\ g \longmapsto g \circ f \end{array} \right) & \text{on morphisms} \end{array}$$

So, what is $g \circ f$?

$V_1 \subseteq k^m$, $V_2 \subseteq k^n$ with coords $x_1, \dots, x_m$; $y_1, \dots, y_n$ resp.

$f: V_1 \longrightarrow V_2$, $f = (f_1, \dots, f_n)$

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
I(V_2) \arrow[r, "(*)"] \arrow[d] & I(V_1) \arrow[d] \\
k[y_1, \dots, y_n] \arrow[r] \arrow[d] & k[x_1, \dots, x_m] \arrow[d] \\
y_i \arrow[r, mapsto] & f_i \\
g \in k[V_2] \arrow[r, dashed, "\exists! \bar{\varphi} = f^*"] \arrow[d] & k[V_1] \arrow[d] \\
0 & 0
\end{tikzcd}$$

$(*)$ $g \in I(V_2) \Rightarrow g(f(Q)) = 0 \quad \forall Q \in V_1$

$\Rightarrow g(f_1, \dots, f_m) = \varphi(g)$ vanishes on V_1

$\Rightarrow \varphi(g) \in I(V_1)$

Ex.

$y - x^2 = 0$ $\quad f(x,y) = x + 5(y - x^2)$

$\varphi: k[x] \to k[x,y]$

$x \mapsto x + 5(y - x^2)$

PF OF COR: Essential Surjectivity: R a reduced f.g. k-algebra $\Rightarrow R \cong k[x_1, \dots, x_n]/_I$

I is radical since R is reduced. ✓

Faithfulness: Given $\bar{\varphi}$, lift to $\varphi: k[y_1, \dots, y_n] \to k[x_1, \dots, x_n]$ well-def'd up to elts in $I(V_1)$ $\Rightarrow$ f only depends on $\bar{\varphi}$. ✓

Fullness: Basically the same as faithfulness. ✓

Projective spaces

Def. $\mathbb{P}^n_k := (k^{n+1} \setminus \{0\})/\sim$ where $(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n)$ $\forall \lambda \in k^\times$.

Equivalence classes: $[x_0 : \dots : x_n]$, homogeneous coordinates.

Rmk. 1) $\underset{\text{point}}{[x_0 : \dots : x_n] \in \mathbb{P}^n} \longleftrightarrow \{(\lambda x_0, \dots, \lambda x_n) \mid \lambda \in k\} \subseteq k^{n+1}$ line through 0

2) $U_i := \{[x_0 : \dots : x_n] \mid x_i \neq 0\} \xrightarrow{\sim} k^n$

$$[x_0 : \dots : x_n] \longmapsto \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}\right)$$

Intuitively: $\mathbb{P}^n_k \cong U_0 \amalg \underbrace{(\text{points at } \infty)}_{x_0 = 0} \cong k^n \amalg \mathbb{P}^{n-1}_k$

Correspondence: let $f \in k[x_0, \dots, x_n]$ homogeneous $\Rightarrow$ $f(x_0, \dots, x_n) \overset{?}{=} 0$ depends only on $[x_0, \dots, x_n]$, $V(f) = \{[x_0, \dots, x_n] \in \mathbb{P}^n \mid f(x_0, \dots, x_n) = 0\}$ makes sense

For a homog. ideal I, $V(I) = \{x \in \mathbb{P}^n \mid f(x) = 0 \ \forall f \in I \text{ homog.}\}$

Rmks. a) The $V(I)$ form closed sets of a topology $\longrightarrow$ Zariski top.

b) Projective variety := irreducible projective algebraic set

Conversely: for $V \subseteq \mathbb{P}^n$, $I(V) = \left\{ \begin{array}{l} \text{ideal gen'd by homog. polys in } k[x_0,\dots,x_n] \\ \text{which vanish on } V \end{array} \right\}$ homog. ideal

Thm. There is a 1:1 correspondence

$$\{\text{alg. sets in } \mathbb{P}^n_k\} \underset{V}{\overset{I}{\rightleftarrows}} \left\{ \begin{array}{l} \text{radical homog. ideals in } k[x_0,\dots,x_n] \\ \text{except } (x_0,\dots,x_n) \end{array} \right\}$$

Pf: Idea: reduce it to the corresp. for k^{n+1}.

$\mathfrak{a} \subset k[x_0,\dots,x_n]$ homog. $\longmapsto$ $V(\mathfrak{a}) \subseteq \mathbb{P}^n$, as before

$\tilde{V}(\mathfrak{a}) := \{x \in k^{n+1} \mid f(x) = 0 \ \forall f \in \mathfrak{a}\}$

Relation b/w $V(\mathfrak{a})$ and $\tilde{V}(\mathfrak{a})$? 3 cases:

- $V(\mathfrak{a}) = \emptyset$ & $\tilde{V}(\mathfrak{a}) = \{0\}$ $\Rightarrow I(\tilde{V}(\mathfrak{a})) = (x_0,\dots,x_n) = \sqrt{\mathfrak{a}}$, excluded
- $V(\mathfrak{a}) = \emptyset$ & $\tilde{V}(\mathfrak{a}) = \emptyset$ $\Rightarrow I(V(\mathfrak{a})) = k[x_0,\dots,x_n] = I(\tilde{V}(\mathfrak{a}))$
- $V(\mathfrak{a}) \neq \emptyset \Rightarrow \tilde{V}(\mathfrak{a}) \neq \emptyset$. Then:
  (i) f homog. vanishes on $\tilde{V}(\mathfrak{a})$ iff vanishes on $V(\mathfrak{a})$ (by def.)
  (ii) $I(\tilde{V}(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ is homog. (exercise sheet 1)

$\Rightarrow \underset{(i)+(ii)}{I(V(\mathfrak{a})) = I(\tilde{V}(\mathfrak{a}))} \underset{\text{HNS}}{=} \sqrt{\mathfrak{a}}$ $\Rightarrow$ $I \circ V|_{\text{homog. ideals except } (x_0,\dots,x_n)} = \mathrm{id}$

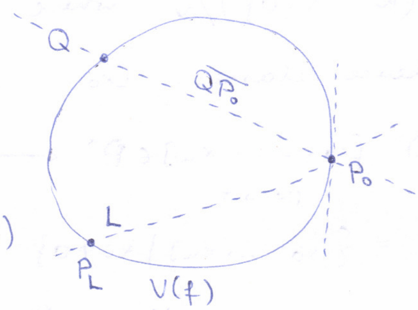
Hence we get the bijection using that V is surjective. □

Morphisms of projective algebraic sets

Ex. f irred, $\deg f = 2$, $f \in k[x,y,z]$ $\Rightarrow V(f) \subseteq \mathbb{P}^2_k$ conic hypersurface

$\exists\ V(f) \longrightarrow \mathbb{P}^1_k$ bijection, specifically $Q \mapsto QP_0$

$\mathbb{P}^n_k = \{\text{lines through a pt in } k^{n+1}\}$



$\Rightarrow$ the claimed map is a bijection if we map $P \longmapsto$ (tangent at P_0, whatever that is)

In the other direction: $V(f) \longleftarrow \mathbb{P}^1$

$P_L \longleftarrow L$ line through P_0

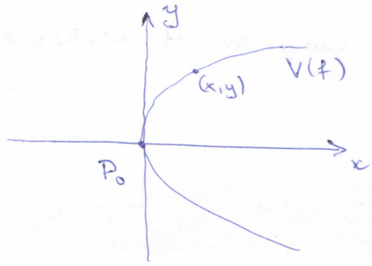
$f|_L$ vanishes at 2 pts; counted with multiplicity: $V(f|_L) = \{P_0, P_L\}$

This should be a morphism of proj. alg. sets.

Naïve approach: do what we did for affines. This will fail for the above example.

More concretely, let $f := xz - y^2$, $P_0 = [0,0,1]$, $U := \{z \neq 0\} \cong k^2 \cong \{z = 1\}$

In chart U: $f = 0$ iff $x - y^2 = 0$



$F(x,y) = [x:y] \in \mathbb{P}^1_k$

$F([x:y:z]) = [x:y]$

Problem: F does the job but not def'd at $P_0 = [0:0:1]$.

Second try: $[x:y] = \left[\frac{x}{y}, 1\right] = \left[\frac{y}{z}, 1\right] = [y:z]$, so try $F([x:y:z]) = [y:z]$

for $y \neq 0 \neq z$, $xz - y^2 = 0 \Rightarrow \frac{x}{y} = \left(\frac{z}{y}\right)^{-1}$

The problem is the same: F is now not def'd at $[1:0:0] \in V(f)$.

One can show that the naive approach will always fail. More precisely:

<u>Fact</u>. There is no non-constant globally defined morphism $\mathbb{P}^2 \to \mathbb{P}^1$.

<u>Upshot</u>. Define morphisms locally, e.g. on sets like $\{y \neq 0\}$ or $\{z \neq 0\}$ then glue them where the sets overlap, e.g. $\{yz \neq 0\}$. That is, we need to work with <u>sheaves</u>.

Sheaves

X a topological space.

<u>Def</u>. A <u>presheaf</u> of sets on X is a contravariant functor $\mathcal{F}: \mathcal{T}^{op} \to \mathrm{Set}$, where $\mathrm{Ob}\,\mathcal{T} := \{U \subseteq X \text{ open}\}$, $\mathrm{Mor}\,\mathcal{T} := \{U \hookrightarrow V \text{ inclusion}\}$.

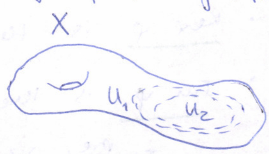
i.e. $\forall U \subseteq X$ we have a set $\mathcal{F}(U)$,

$\forall U_2 \subseteq U_1$ a map $\mathcal{F}(U_1) \xrightarrow{r_{U_2,U_1}} \mathcal{F}(U_2)$. (Notation: $r_{U_2,U_1}(s) =: s|_{U_2}$)

s.t. $r_{U,U} = \mathrm{id}_U$, $\forall U_3 \subseteq U_2 \subseteq U_1$: $r_{U_3,U_1} = r_{U_3,U_2} \circ r_{U_2,U_1}$.

<u>Ex</u>. 1) $\mathcal{A}(U) = \mathbb{Q}$ $\forall U$, $\forall r_{U_2,U_1} = \mathrm{id}_{\mathbb{Q}}$. $\mathcal{A}$ is a presheaf of sets/groups/rings/fields.

2) $X = \mathbb{R}^n$ or a manifold, $C^0(U) = \{\text{cont. functions } U \to \mathbb{R}\}$,
$C^\infty(U) = \{\text{smooth } U \to \mathbb{R}\}$



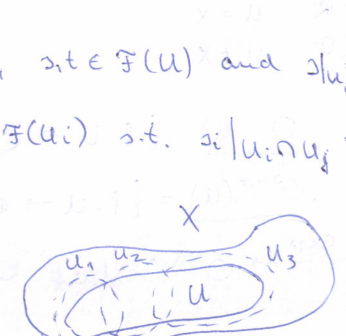
r are restrictions of functions: $r_{U_2,U_1}: C^\infty(U) \to C^\infty(V)$
$f \mapsto f|_V$

Then C^0 and C^∞ are presheaves of rings.

<u>Def</u>. A presheaf $\mathcal{F}$ is a <u>sheaf</u> if

(1) Uniqueness: $U \subseteq X$ open, $\{U_i\}$ open cover of U, $s,t \in \mathcal{F}(U)$ and $s|_{U_i} = t|_{U_i}$ $\forall i \Rightarrow s = t$

(2) Gluing: $U \subseteq X$ open, $\{U_i\}$ open cover of U, $s_i \in \mathcal{F}(U_i)$ s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ $\forall i \neq j \Rightarrow \exists s \in \mathcal{F}(U)$ s.t. $s|_{U_i} = s_i$ $\forall i$.



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Def. A <u>morphism of presheaves</u> is a natural transformation of functors $\mathcal{T}^{op} \to \mathrm{Set}$ where $\mathcal{T}$ is as before. That is, $\mathcal{F} \to \mathcal{G}$ is a morphism of presheaves if $\forall U$ open $\exists \varphi(U)$ s.t. $\forall V \subseteq U$ open:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d] \arrow[dr, phantom, "G"] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r] & \mathcal{G}(V) \end{tikzcd}$$

Ex. <u>Constant presheaf</u> $C_R(U) := R \quad \forall U$.

Not a sheaf: $C_R(\emptyset) = R$, contradicts uniqueness if $\# R \geq 2$.

Ex. <u>Constant sheaf</u>: $\underline{R}(U) = \{\text{locally constant functions } U \to R\}$

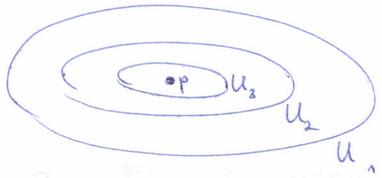
Prop. (Sheafification) $\mathcal{F}$ a presheaf. Then there is a sheaf $\mathcal{F}^*$ and a morphism of presheaves $\vartheta: \mathcal{F} \to \mathcal{F}^*$ universal wrt maps from $\mathcal{F}$ to sheaves, i.e.

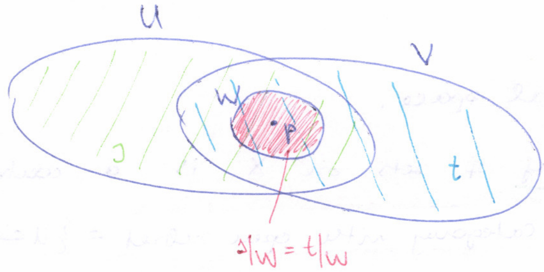
$$\begin{tikzcd} \mathcal{F} \arrow[r, "\forall"] \arrow[d, "\vartheta"'] & \mathcal{G} \in \mathrm{Shv} \\ \mathcal{F}^* \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Moreover, $(\mathcal{F}^*, \vartheta)$ are unique up to unique iso.

Pf: $\underset{\text{stalk}}{\mathcal{F}_p} := \varinjlim_{p \in U} \mathcal{F}(U) = \{(U,s) \mid p \in U, s \in \mathcal{F}(U)\} \big/ \sim$ where $(U,s) \sim (V,t)$ if $\exists W \subseteq U \cap V$, $s|_W = t|_W$.

germs of functions





$$\mathcal{F}^*(U) := \left\{ f: U \to \bigcup_{p \in U} \mathcal{F}_p \;\middle|\; \begin{array}{l} \bullet\ \forall p \in U: \ f(p) \in \mathcal{F}_p \ \& \\ \bullet\ \forall p \in U\ \exists W \subseteq U \text{ open},\ p \in W,\ \exists s \in \mathcal{F}(W): \\ \qquad \forall q \in W: f(q) = s|_{\mathcal{F}_q} \end{array} \right\}$$

$\square$

Def $\varphi: \mathcal{F} \to \mathcal{G}$ sheaves of ab gps.

- <u>$\operatorname{Ker} \varphi$</u> is the sheaf def'd by $(\operatorname{Ker} \varphi)(U) := \operatorname{Ker}\big(\varphi(U): \mathcal{F}(U) \to \mathcal{G}(U)\big)$
- <u>$\operatorname{Im} \varphi$</u> is the sheafification of the image presheaf:

  $\operatorname{Im}^{pre}(\varphi)(U) := \operatorname{Im}\big(\varphi(U): \mathcal{F}(U) \to \mathcal{G}(U)\big)$.

Ex. The sheaf associated to C_R is $\underline{R}$.

Ex. $\mathcal{F}(U) = \begin{cases} R & U = X \\ 0 & U \neq X \end{cases}$

Ex. $X = \mathbb{C}$.

$$\left.\begin{array}{l} \underline{\mathcal{O}^{an}}(U) = \{f: U \to \mathbb{C} \text{ holomorphic}\} \\ \underline{\mathcal{O}^{an,\times}}(U) = \{f: U \to \mathbb{C} \text{ holomorphic, nowhere vanishing}\} \end{array}\right\} \text{sheaves}$$

$\mathcal{O}^{an} \xrightarrow{\exp} \mathcal{O}^{an,\times}$ sheaf homomorphism, $0 \to \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O}^{an} \xrightarrow{\exp} \underbrace{\mathcal{O}^{an,\times} \to 0}_{\text{locally there is a log}}$

But $0 \to \mathbb{Z} \to \mathcal{O}^{an} \to \mathcal{O}^{an,\times} \to \mathbb{Z} \to 0$ (we took $U := \mathbb{C}\setminus\{0\}$)

<u>Def.</u> $f: X \to Y$ continuous, $\mathcal{F}$ a sheaf on X. Pushforward sheaf $f_*\mathcal{F}$ def'd as
$f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U))$.

Restrictions:

$$\begin{array}{ccc} f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)) \\ \downarrow r_{VU} \qquad \vdots \\ f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V)) \end{array} \qquad \text{for } V \subseteq U,\ f^{-1}(V) \subseteq f^{-1}(U).$$

Structure sheaf.

<u>Problem.</u> How to define morphisms for non-affine alg varieties/sets?

<u>Key example.</u> M, N C^∞-mf., $f: M \to N$ cont.

<u>Claim.</u> f smooth iff the pullback of every smooth function is smooth,
i.e. $\forall V \subseteq N$ open $\forall V \to \mathbb{R}$ smooth: $f_* g = g\circ f$ is smooth

$\Leftarrow$: $f = (f_1, \dots, f_n)$: $\underset{\mathbb{R}^m}{U} \xrightarrow{f} \underset{\mathbb{R}^n}{V} \xrightarrow{y_i} \mathbb{R}$ ($U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$)

RHS: $\underbrace{f_* y_i}_{f_i}$ is smooth $\forall i$ $\Rightarrow f = (f_1, \dots, f_n)$ smooth

<u>Def.</u> $X \subseteq k^n$, $U \subseteq X$ open, $f: U \to k$ is <u>regular</u> if $\forall x \in U$ $\exists V \subseteq U$ open: $g, h \in k[x_1, \dots, x_n]$ s.t. h vanishes nowhere on V and $f = \frac{g}{h}$. on V.

<u>Def.</u> <u>Structure sheaf</u> $\mathcal{O}_X(U) = \{f: U \to k \text{ regular}\}$, the restriction maps are restrictions of functions.

<u>(Re)Def.</u> <u>Affine variety</u>: top. space X and $\mathcal{O}_X$ sheaf of k-val functions, $(X, \mathcal{O}_X) \simeq (Y, \mathcal{O}_Y)$ for some irr'ble aff alg set Y with structure sheaf $\mathcal{O}_Y$.
Here $\simeq$ is understood as a homeo $\varphi: X \xrightarrow{\cong} Y$ with $\varphi^*: \mathcal{O}_Y \xrightarrow{\cong} \varphi_* \mathcal{O}_X$ iso.

<u>Ex.</u> $\mathbb{A}^n = (k^n, \mathcal{O}_{k^n})$

<u>Def.</u> <u>Prevariety</u>: X top space, $\mathcal{O}_X$ sheaf of k-val functions, X connected, $\exists \{U_i\}$ finite open cover of X s.t. $\forall (U_i, \mathcal{O}_X|_{U_i})$ is an affine variety.

Here $\mathcal{F}|_U$ is def'd by $\mathcal{F}|_U(V) = \mathcal{F}(V)$ $\forall V \subseteq U$. where $U \subseteq X$ open.

<u>Def.</u> $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ prevarieties. $f: X \to Y$ is a <u>morphism</u> if f is continuous and induces

$$\begin{array}{ccc} \mathcal{O}_Y & \xrightarrow{f^*} & f_*\mathcal{O}_X \\ g & \longmapsto & g\circ f \end{array}$$

Prop. $(X, \mathcal{O}_X)$ prevariety iff

(1) $\forall p \in X \ \exists U \subseteq X$: $(U, \mathcal{O}_X|_U) \simeq (Y, \mathcal{O}_Y)$ for an aff alg set Y.

(2) X irr'ble noetherian top space.

Def. X top space noetherian if any descending chain of closed subsets stabilises.

Pf: • Assume (1–2).

X irr'ble $\Rightarrow$ connected.

X irr'ble $\Rightarrow$ every nonempty open is dense and irr'ble. Noetherianity $\Rightarrow$ finite covering.

• Conversely: topological argument.

□

Now: 1) we recover the def of morphism of aff alg sets

2) open subset in a prevariety is a prevariety.

Prop. X aff alg set in k^n $\Rightarrow$ $\underbrace{\mathcal{O}_X(X)}_{\substack{\text{ring of reg} \\ \text{functions } X \to k}} = \underbrace{k[X]}_{\substack{\text{coord. ring} \\ k[x_1,\dots,x_n]/I(X)}}$

Pf:
$$k[x_1,\dots,x_n] \xrightarrow{\varphi} \mathcal{O}_X(X)$$
$$f \longmapsto (x \longmapsto f(x))$$

$\ker \varphi = I(X)$ $\Rightarrow$ $k[X] \longrightarrow \mathcal{O}_X(X)$ injective

Surjectivity: let $f \in \mathcal{O}_X(X)$. $\Rightarrow \exists p \in X$, $\exists g_p, h_p \in k[x_1,\dots,x_n]$ s.t. $f = \frac{g}{h}$

on some open nbhd W_p of p, $V(h) \cap W = \emptyset$.

$D(h) := \{x \in X \mid h(x) \neq 0\}$ forms a basis of the Zariski topology. (exc.)

Step 1. $\forall p \in X \ \exists r_p \in k[x_1,\dots,x_n]$ s.t. $D(r_p) \subset W_p$.

On $D(r_p)$: $\quad f = \frac{g_p}{h_p} = \frac{g_p \cdot r_p}{h_p \cdot r_p} =: \frac{\tilde{g}_p}{\tilde{h}_p}$ on $D(r_p)$, also on $D(\tilde{h}_p) = D(r_p) \cap D(h_p)$

$\Rightarrow X = \bigcup_p D(\tilde{h}_p)$ s.t. $f = \frac{\tilde{g}_p}{\tilde{h}_p}$ on $D(\tilde{h}_p)$.

Step 2. Finite cover: use partition of unity (and HNS to obtain it)

Step 3. Glue

□

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Note. $f : X \to Y$ is a morphism iff continuous and $f^* : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is a morphism.

For $X = k^n$, $Y = k^n$, $f = (f_1, \dots, f_n)$ morphism, $f_i \in k[X]$: f is indeed continuous,

and $f^*(y_i) = f_i \in k[X]$ $\Rightarrow$ $f^* : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is indeed a morphism

Conversely: $f^*(y_i) =: f_i$, $f^*(y_i) \in \mathcal{O}_X(X)$. So if $\mathcal{O}_X(X) = k[X]$ then f is a morphism in the sense of our defn of morphisms of affine algebraic varieties.

And this is indeed the case in the light of our Prop. from last time.

Thus we have proven that this defn of prevarieties agrees with the def. of morphisms of aff alg sets. (for aff alg. sets, which are, obviously, prevarieties).

Prop: $(X, \mathcal{O}_X)$ prevar., $U \subseteq X$ open $\Rightarrow$ $(U, \mathcal{O}_X|_U)$ prevariety.

Pf: Clear from def.: U is also irred and noetherian.

$\forall p \in U$. $\exists W \subseteq X$ open s.t. $(W, \mathcal{O}_X|_W) \cong$ aff. alg. set

$\Rightarrow$ wma X to be affine, $X \subseteq k^n$

Since $D(f)$ give a basis of the topology on X, $f \in k[X]$,

$\exists f \in k[X]$: $U \supseteq D(f)$. Claim. $(D(f), \mathcal{O}_X|_{D(f)})$ is an affine alg. variety.

To show this, write $X = V(\mathfrak{a})$, $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$

$\mathfrak{a}' := (\mathfrak{a}, x_{n+1} f - 1) \subseteq k[x_1, \dots, x_{n+1}]$

We have a correspondence, even an iso

$$D(f) \underset{\longleftarrow}{\overset{\sim}{\longrightarrow}} V(\mathfrak{a}') \subseteq k^{n+1}$$

$$(x_1, \dots, x_n) \longmapsto \left(x_1, \dots, x_n, \frac{1}{f(x_1, \dots, x_n)}\right)$$

$$(x_1, \dots, x_n) \longleftarrow\!\shortmid (x_1, \dots, x_{n+1})$$

and the structure sheaves $\mathcal{O}_X|_{D(f)}$ and $\mathcal{O}_{V(\mathfrak{a}')}$ match.

Indeed: $f \in \mathcal{O}_{V(\mathfrak{a}')}(U)$ $\Rightarrow$ $f = \frac{g}{h}$ locally $\Rightarrow$ $\varphi^* f = \dfrac{g(x_1, \dots, x_n, \frac{1}{f}) \cdot f^N}{h(x_1, \dots, x_n, \frac{1}{f}) f^N} = \dfrac{\tilde{g}(x_1, \dots, x_n)}{h'(x_1, \dots, x_n)}$

locally for some $N \gg 0$. The converse is similar. □

Projective varieties are prevarieties

$\mathfrak{a} \subset k[x_0, \dots, x_n]$ homogeneous

$X = V(\mathfrak{a}) = \{x \in \mathbb{P}^n \mid f(x) = 0 \ \forall f \in \mathfrak{a}\} \subset \mathbb{P}^n_k$ closed, $U \subseteq X$ open

Def. $f: U \to k$ is regular if $\forall p \in U$ $\exists W \subseteq U$ open, $p \in W$, $g, h \in k[x_0, \dots, x_n]$, $\deg g = \deg h$, $f = \frac{g}{h}$ on W and $W \subseteq D(h)$. (Note that $g(x)$ is not well-def'd but g/h is.)

Structure sheaf: $\mathcal{O}_X(U) := \{f : U \to k \text{ regular}\}$. We call $(X, \mathcal{O}_X)$ a proj. variety.

Prop. X irred. proj. alg. set $\Rightarrow$ $(X, \mathcal{O}_X)$ is a prevariety.

Pf: X irreducible ✓

X noetherian b/c $\mathbb{P}^n$ is ✓

$U_i := \{x_i \neq 0\} \subset \mathbb{P}^n_k$, $U_i \cong k^n$. We check that $U_i \cap X$ is affine, wma $i=0$.

$$X \cap U_0 = \{x \in \mathbb{P}^n_k \mid x_0 \neq 0,\ f(x) = 0\ \forall f \in \mathfrak{a}\}$$

Recall: $U_0 \xrightarrow[\varphi]{\sim} k^n$ homeo.

$$[x_0, \dots, x_n] \longmapsto \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$$

$$f(x_0, \dots, x_n) = 0 \iff \underbrace{x_0^{\deg f}}_{\neq 0} f\left(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) = 0$$

$$\overset{\varphi}{\iff} f(1, y_1, \dots, y_n) = 0$$

$\Rightarrow X \cap U_0 \cong \{(y_1, \dots, y_n) \in k^n \mid f(1, y_1, \dots, y_n) = 0\ \forall f \in \mathcal{O}_X\}$

$\mathfrak{a}' := (f(1, y_1, \dots, y_n) \mid f \in \mathfrak{a})$ ideal, $\varphi : X \cap U_0 \xrightarrow{\sim} V(\mathfrak{a}')$ homeo

Check that φ takes regular functions to regular functions:

$\underset{\text{open}}{W \subset} V(\mathfrak{a}) \cap U_0$, $W \to k$ regular, that is, locally $f = \dfrac{g(x_0, \dots, x_n)}{h(x_0, \dots, x_n)} = \dfrac{x_0^d\, g(1, \frac{x_1}{x_0}, \dots)}{x_0^d\, h(1, \frac{x_1}{x_0}, \dots)}$

$\iff f = \tilde{f} \circ \varphi$. $\tilde{f} = \dfrac{\tilde{g}}{\tilde{h}}$ locally, $\tilde{g} = g(1, y_1, \dots, y_n)$, $\tilde{h} = h(1, y_1, \dots, y_n) \in k[y_1, \dots, y_n]$

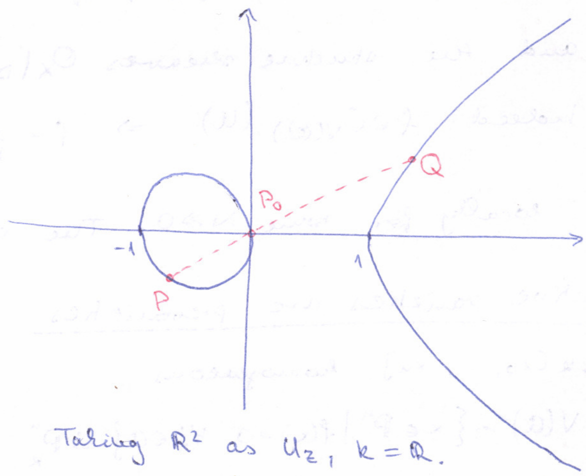
$\iff f = \varphi^* \tilde{f}$ when $\tilde{f}$ is regular on $V(\mathfrak{a}')$.

Conversely: same with $f \circ \varphi^{-1} = \tilde{f}$. $\square$

Ex. $(zy^2 = x(x^2 - z^2)) \subset \mathbb{P}^2_k$. $U_z = \{z \neq 0\}$

$\varphi \downarrow$ $\qquad\qquad$ $\varphi \downarrow$

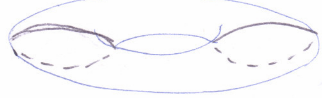
$Y^2 = X(X^2 - 1)$ $\qquad$ $\mathbb{P}^2$, $X = \frac{x}{z}$, $Y = \frac{y}{z}$



Taking $\mathbb{R}^2$ as U_z, $k = \mathbb{R}$.

Let C denote $\{zy^2 = x(x^2 - z^2)\}$.

C



What do we miss? If $z = 0 \Rightarrow x^3 = 0 \Rightarrow y = 1$, $[0, 1, 0]$.

so only the pt at ∞ is missing.

This is a torus, we have 2 circles, and the one on the right has in this section the point ∞ as antipodal to $(1, 0)$.

$\varphi : C \longrightarrow C$

$P \longmapsto Q$, $(L \cap C = \{P, Q, P_0\})$

$P_0 \longmapsto P_\infty$

$P_\infty \longmapsto P_0$

$L = \{(at, bt) \mid t \in k\}$

$L \cap C \longleftrightarrow$ sols of $b^2 t^2 = at(a^2 t^2 - 1)$

$\iff 0 = t(t-1)(a^2 t + 1)$

$\qquad\ \downarrow \quad \downarrow \qquad \downarrow$

$\qquad P_0 \quad P \qquad Q = \left(-\frac{1}{a}, -\frac{b}{a^2}\right)$

Criterion: X, Y prevarieties, $f: X \to Y$ any map, $\{V_i\}$ aff open cover of Y, $\{U_i\}$ open cover of X s.t. $f(U_i) = V_i$, $f^*\mathcal{O}_Y(V_i) \xrightarrow[(f|_{U_i})^*]{} \mathcal{O}_X(U_i)$ induced.

$\Rightarrow$ f is a morphism.

Pf next lecture.

Application: $V_1 := U_z \cong k^2$, $V_2 := U_y$.

$f^{-1}(V_1) = C \setminus P_0$, $f^{-1}(V_1) = \underbrace{(U_z \cap U_x)}_{V_1} \cup \underbrace{U_y}_{V_2}$

Rmk. $\mathcal{O}_X(X) = k[X]$

locally def'd — globally def'd

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E.g. $k = \mathbb{R}$, $X = \mathbb{A}^2_{\mathbb{R}}$. $f = \frac{1}{x^2+y^2+1} \notin \mathbb{R}[\mathbb{A}^2_{\mathbb{R}}] = \mathbb{R}[x,y]$

PF OF CRIT: f morph. $\Leftrightarrow$ continuous & f^* induces $\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$.

Step 1. $U_i \subset X$ open $\Rightarrow$ U_i is covered by open affines $U_i = \bigcup_j U_{ij}$

$f(U_{ij}) \subset V_i$, $g \in \mathcal{O}_Y(V_i)$ $\Rightarrow$ $(f|_{U_{ij}})^*(g) = (f|_{U_i})^*(g)|_{U_{ij}}$

Thus we get $\mathcal{O}_Y(V_i) \xrightarrow{f|_{U_{ij}}^*} \mathcal{O}_X(U_{ij})$. Replace $\{(U_i, V_i)\}$ by $\{(U_{ij}, V_i)\}$

$\Rightarrow$ WMA U_i to be affine.

Step 2.
$$\begin{array}{ccc} U_i & \xrightarrow{F = f|_{U_i}} & V_i \\ \int \text{closed} & & \int \text{closed} \\ k^m & & k^n \end{array}$$

By assumption: $F^*: \mathcal{O}_{V_i}(V_i) \longrightarrow \mathcal{O}_{U_i}(U_i)$, $\mathcal{O}_{V_i}(V_i) = k[y_1,\dots,y_n]/I(V_i)$, $\mathcal{O}_{U_i}(U_i) = k[x_1,\dots,x_m]/I(U_i)$

Hence F is locally given by polynomials:

$\bar{F}_\ell := F^*(y_\ell)$, lift $\bar{F}_\ell$ to $F_\ell \in k[x_1,\dots,x_m]$, then $F = (F_1,\dots,F_n)$,

thus F is a morphism of aff alg sets, thus F is a morphism of prevarieties.

Step 3. X covered by U_i s.t. $f|_{U_i}$ is a morphism.

Checking that f is a morphism:

- continuity: $W \subseteq Y$ open, $\Rightarrow$ $f^{-1}(W) = \bigcup_i \underbrace{f^{-1}(W) \cap U_i}_{= f|_{U_i}^{-1}(W \cap V_i) \text{ open}}$ $\Rightarrow$ $f^{-1}(W)$ open ✓
- $g \in \mathcal{O}_Y(W)$, $f^*(g): f^{-1}(W) \to k$

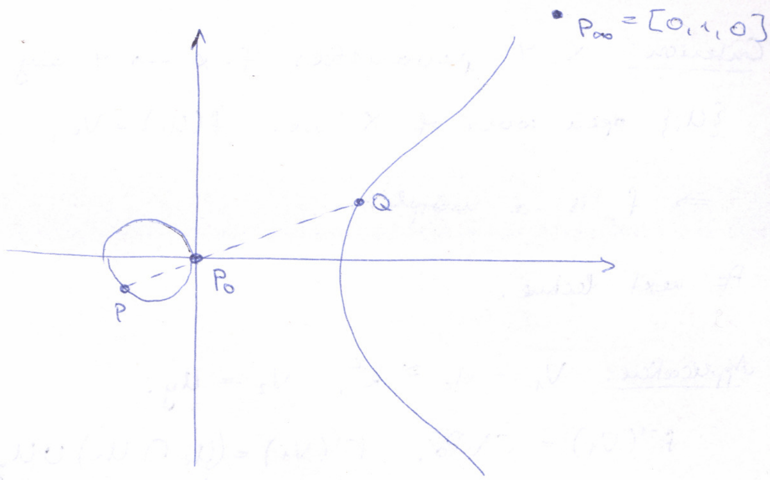
$f^*(g)|_{U_i \cap f^{-1}(W)} = \underbrace{(f|_{U_i})^*(\underbrace{g|_{V_i \cap W}})}_{\in \mathcal{O}_X(U_i \cap f^{-1}(W)) \text{ by Step 2.}}$

$g|_{V_i}: V_i \cap W \to k$, $g|_{V_i} \in \mathcal{O}_Y(V_i \cap W)$

$\Rightarrow$ $f^*(g) \in \mathcal{O}_X(f^{-1}(W))$ since $\mathcal{O}_X$ is a sheaf.

$\Rightarrow f^*(g) \in \mathcal{O}_X(f^{-1}(W))$ since $\mathcal{O}_X$ is a sheaf.

Ex. $\left(zy^2 = x(x^2 - z^2)\right) \subset \mathbb{P}^2_k$



$U_z = \{z \neq 0\} \simeq k^2$

$X = \frac{x}{z}, \quad Y = \frac{y}{z}$

$$\varphi: \begin{array}{ccc} C & \longrightarrow & C \\ P & \longmapsto & Q \\ \infty & \longmapsto & 0 \\ 0 & \longmapsto & \infty \end{array}$$

$(a,b) \longmapsto \left(-\frac{1}{a}, -\frac{b}{a^2}\right)$ for $(a,b) \in k^2 \setminus \{0\} \simeq U_z \setminus P_0$

$U_1 := U_z \cap U_x \cap C = C \setminus \{P_0, P_\infty\} \xrightarrow{\varphi} V_1 = U_z \cap C$

$U_2 := U_y \cap C = C \setminus \{P_0, (-1,0), (1,0)\} \xrightarrow{\varphi} V_2 = U_z \cap C$

$U_3 := C \setminus \{\infty, (1,0), (-1,0)\} \xrightarrow{\varphi} V_3 = U_y \cap C$

all open and cover C — affine opens

$\varphi|_{U_1}: U_1 \longrightarrow V_1$ (X, Y) $\qquad \mathcal{O}_C(U_z \cap C) = k[X,Y] / (Y^2 = X(X^2-1))$

$\varphi|_{U_1}^*(X) = -\frac{1}{X}$

$\varphi|_{U_1}^*(Y) = -\frac{Y}{X^2} \qquad \longrightarrow$ regularity on U_1 ✓

$\varphi|_{U_2}: U_2 \longrightarrow V_2$

$S = \frac{x}{y}, \quad T = \frac{z}{y} \qquad X, Y$

$[(c,1,d)] \in U_2$

$\Downarrow$

$\varphi([(c,1,d)]) = \varphi\left(\left[\frac{c}{d}, \frac{1}{d}, 1\right]\right) = \left[-\frac{d}{c}, -\frac{d^2}{c^2}\cdot\frac{1}{d}, 1\right]$

$= \left[-\frac{d}{c}, -\frac{d}{c^2}, 1\right]$

$\Rightarrow$ in U_2, $\varphi(c,d) = \left(-\frac{d}{c}, -\frac{d}{c^2}\right)$

This is not def'd when $c=0$.

$T = S(S^2 - T^2) \Rightarrow d = c(c^2 - d^2) \Rightarrow \varphi(c,d) = \left(-\frac{d}{c}, -\frac{d}{c^2}\right)$

$= \big(\underbrace{-(c^2-d^2)}_{X(\varphi(c,d)) = \varphi^* X(c,d)}, -c + d(c^2 - d^2)\big)$

This defines φ on the whole U_2.

$P_\infty = [0,1,0] = (0,0) \in U_2$ gets sent to $\varphi(P_\infty) = (0,0) \in U_z$.

$\Rightarrow \varphi^* X = -(S^2 - T^2), \quad \varphi^* Y = -S + T(S^2 - T^2)$, these are regular

$\Rightarrow \varphi|_{U_2}$ is regular ✓

The last chart U_3 is left to the reader.

Rmk. $zy^2 = x(x-1)(x-\lambda)$ for some $\lambda \in k$.

$\lambda = -1$ as above, $\lambda = 1$ quite different: 

Products

Def. $\mathcal{C}$ a category, $X, Y \in \mathrm{Ob}\,\mathcal{C}$, then $Z \in \mathrm{Ob}\,\mathcal{C}$ with $(p: Z \to X), (q: Z \to Y) \in \mathrm{Mor}\,\mathcal{C}$ is a <u>product</u> of X and Y if $\forall W \in \mathrm{Ob}\,\mathcal{C}$ $\forall (f: W \to X)\ (g: W \to Y) \in \mathrm{Mor}\,\mathcal{C}$ $\exists! (t: W \to Z) \in \mathrm{Mor}\,\mathcal{C}$ s.t. $p \circ t = f$, $q \circ t = g$.

$$\begin{tikzcd} & & X \\ \forall W \arrow[urr, "\forall f"] \arrow[rr, dashed, "\exists! t"] \arrow[drr, "\forall g"'] & & Z \arrow[u, "p"'] \arrow[d, "q"] \\ & & Y \end{tikzcd}$$

If Z exists then it is unique (up to unique iso).

$$\mathrm{Hom}_{\mathcal{C}}(W, Z) = \mathrm{Hom}_{\mathcal{C}}(W, X) \times \mathrm{Hom}_{\mathcal{C}}(W, Y)$$

Thm. The category of prevarieties has products.

Let X be a prevariety.

$\mathrm{Hom}_{\mathrm{PreVar}}(\mathbb{A}^0, X) = X$

If $X \times Y$ exists then $|X \times Y| = \mathrm{Hom}(\mathbb{A}^0, X \times Y) = \mathrm{Hom}(\mathbb{A}^0, X) \times \mathrm{Hom}(\mathbb{A}^0, Y) = |X| \times |Y|$ where $|X|$ denotes the underlying set.

Thus we need to endow $|X| \times |Y|$ with a topology and structure sheaf.

Prop. X, Y aff. varieties. Then

a) $\exists X \times Y$ product prevariety.

b) $X \times Y$ is affine with coordinate ring $k[X \times Y] = k[X] \otimes_k k[Y]$.

Pf: $X \subseteq k^m$, $Y \subseteq k^n$, $X = V(f_1, \dots, f_\ell)$, $Y = V(g_1, \dots, g_q)$

$\Rightarrow$ $X \times Y \subseteq k^{m+n}$ as a set is the zero locus of $f_1, \dots, f_\ell, g_1, \dots, g_q$.

So define $X \times Y$ as a prevariety as $V(f_1, \dots, g_1, \dots)$

$$k[x_1, \dots, y_1, \dots] \big/ (f_1, \dots, g_1, \dots) \simeq k[x_1, \dots] \big/ (f_1, \dots) \otimes_k k[y_1, \dots] \big/ (g_1, \dots) \simeq k[X] \otimes_k k[Y]$$

As X is an aff var., $k[X]$ is an integral domain. Same for $k[Y]$ $\Rightarrow$

$\Rightarrow$ so is $k[X] \otimes_k k[Y]$. $\Rightarrow$ $(f_1, \dots, g_1, \dots)$ is a radical prime ideal, hence this is $I(X \times Y)$, $k[X \times Y] = k[X] \otimes k[Y]$.

$X \times Y$ def'd by prime ideal $\Rightarrow$ irr'ble $\Rightarrow$ aff vty.

Checking that $X \times Y$ is a category theoretical product:

Use the Criterion for $U_i = W$, $V_i = X \times Y$

$k[X \times Y]$ is gen'd by $p^* k[X]$ and $q^* k[Y]$, and $t^* p^* k[X] = f^* k[X] \subseteq \mathcal{O}_W(W)$

We skipped a few details here, e.g. didn't check that everything is a morphism.

PF OF THM: X, Y prevars

Step 1. Endow $|X| \times |Y|$ with topology and structure sheaf

Topology: define basis of topology by

$$\left\{ W \subseteq X \times Y \;\middle|\; \begin{array}{l} \exists U \subseteq X \text{ affine } \exists V \subseteq Y \text{ affine s.t. } W \subseteq U \times V \text{ and} \\ W = D(\sum_i f_i g_i) \text{ for } f_i \in k[U],\ g_i \in k[V] \end{array} \right\} =: \mathcal{B}$$

Let $D(\sum f_i g_i) \subseteq U \times V$, $D(\sum f'_j g'_j) \subseteq U' \times V'$.

$\Rightarrow D(\sum f_i g_i) \cap D(\sum f'_j g'_j) \subseteq (U \cap U') \times (V \cap V')$

$D(\sum f_i g_i) \cap D(\sum f'_j g'_j) \cap U''_i \times V''_i = D\left((\sum f_i g_i)|_{U''_i \times V''_i} \cdot (\sum f'_j g'_j)|_{U''_i \times V''_i}\right) \in \mathcal{B}$

Structure sheaf: $\mathcal{O}_{X\times Y}(W) := \mathcal{O}_{U \times V}(W)$, this $\mathcal{O}_{X \times Y}$ is a sheaf wrt. $\mathcal{B}$

Let $\mathcal{O}_{X\times Y}$ be the induced sheaf on $X \times Y$ (to be explained in lecture 6)

Check $(X \times Y, \mathcal{O}_{X\times Y})$ is a categorical product

$X \times Y$ is a prevariety: connected and covered by the aff vtys $U \times V$

Universal property: $f: W \to X$, $g: W \to Y$. We have a map of sets $t: W \to X \times Y$.

Use the Crit: cover $X \times Y$ by $U \times V$.

$t^{-1}(U \times V) = f^{-1}(U) \cap g^{-1}(V)$

$$\begin{tikzcd} & U \\ t^{-1}(U\times V) \arrow[ur, "f"] \arrow[r] \arrow[dr, "g"'] & U \times V \arrow[u] \arrow[d] \\ & V \end{tikzcd}$$

$\Rightarrow t|_{t^{-1}(U\times V)}$ is the unique map from

$f|_{t^{-1}(U\times V)}$ and $g|_{t^{-1}(U \times V)}$ $\Rightarrow t|_{t^{-1}(U\times V)}$ is a morphism by Prop. $\square$

29.10.2018

Rmk. Basis of topology.

For an aff var X, $\{D(f) \mid f \in k[X]\}$ is a basis

Def. $\mathcal{F}$ sheaf wrt $\mathcal{B}$: $\forall U \in \mathcal{B}$ a set $\mathcal{F}(U)$, $\forall U \subset V \in \mathcal{B}$ restrictions $\mathcal{F}(V) \to \mathcal{F}(U)$ that satisfy gluing & uniqueness wrt $\mathcal{B}$: $\forall U = \bigcup_i U_i$ cover where $U, U_i \in \mathcal{B}$:

I) $s, t \in \mathcal{F}(U)$, $s|_{U_i} = t|_{U_i}\ \forall i \Rightarrow s = t$

II) $s_i \in \mathcal{F}(U_i)$, $\forall i,j\ \exists \{W_{ijk}\}_k$ cover of $U_i \cap U_j$, $W_{ijk} \in \mathcal{B}$, $s_i|_{W_{ijk}} = s_j|_{W_{ijk}}$

$\Rightarrow \exists s \in \mathcal{F}(U)$, $s_i = s|_{U_i}$. | Stalk: $\mathcal{F}_p = \varinjlim_{p \in U \in \mathcal{B}} \mathcal{F}(U)$

Prop: $\mathcal{F}$ sheaf on $\mathcal{B} \Rightarrow \exists! \mathcal{F}'$ sheaf on X, $\mathcal{F}(U) = \mathcal{F}'(U)\ \forall U \in \mathcal{B}$.

PF: Like sheafification.

$$\mathcal{F}'(W) := \left\{ s: W \to \bigcup_{p \in W} \mathcal{F}_p \;\middle|\; \begin{array}{l} s(p) \in \mathcal{F}_p\ \forall p \in W \text{ and} \\ \forall p \in W\ \exists V \in \mathcal{B},\ p \in V,\ \exists t \in \mathcal{F}(V)\ \forall q \in V\ t_q = s(q) \in \mathcal{F}_q \end{array} \right\}$$

Check: this is a sheaf. By gluing & uniqueness $\Rightarrow \mathcal{F}(U) = \mathcal{F}'(U)$.

Uniqueness: prove univ prop for maps to sheaves on X.

Ex. X aff alg set, $\mathcal{B} = \{D(f) \mid f \in k[X]\}$, $\mathcal{O}_X$ str shf

$\mathcal{O}_X(X) = k[X] =: R \rightsquigarrow \mathcal{O}_X(D(f)) = k[X]_f$, as we now prove:

$X = V(\mathfrak{a})$, $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$

$X' = V(\mathfrak{a}, x_{n+1} f - 1) \subseteq k^{n+1}$

$\Rightarrow \left(D(f), \mathcal{O}_X|_{D(f)}\right) \cong \left(X', \mathcal{O}_{X'}\right)$

$D(f) \longleftrightarrow X'$

$(x_1, \dots, x_n) \longmapsto (x_1, \dots, x_n, \frac{1}{f})$

$(x_1, \dots, x_n) \longleftarrow (x_1, \dots, x_{n+1})$

$\Rightarrow \mathcal{O}_{X'}(D(f)) = \mathcal{O}_{X'}(X') = k[x_1, \dots, x_n]/(\mathfrak{a}, x_{n+1} f - 1) = k[X][x_{n+1}]/(x_{n+1} f - 1) = k[X]_f$

Alternative construction of $\mathcal{O}_X$:

$\mathcal{O}_X(D(f)) = R_f$, $R := k[X]$. Then def. $\mathcal{O}_X$ on X by Prop, gluing.

Rmk. 1) If $U \subset X$ open $\Rightarrow U \times Y \subset X \times Y$ open

(Cover U by aff opens U_i, Y by aff opens $V_j \Rightarrow U_i \times V_j$ aff open cover of $U \times Y$)

2) $Z \subset X$ closed $\Rightarrow Z \times Y \subset X \times Y$ closed.

Prop. X, Y proj var $\Rightarrow X \times Y$ proj var.

Pf: (Recall: X proj var $\Leftrightarrow X \subseteq \mathbb{P}^n$ irred proj alg set, $\mathcal{O}_X$ structure sheaf)

$X \subseteq \mathbb{P}^m$, $Y \subseteq \mathbb{P}^n$ irred closed

$X \times Y = (X \times \mathbb{P}^n) \cap (\mathbb{P}^m \times Y) \subseteq \mathbb{P}^n \times \mathbb{P}^m$ closed irreducible

$\Rightarrow$ Sts $\mathbb{P}^m \times \mathbb{P}^n$ is a proj var.

Segre embedding: $\mathbb{P}^m \times \mathbb{P}^n \xrightarrow{\varphi} \mathbb{P}^{(m+1)(n+1)-1}$

$([x_1, \dots], [y_1, \dots]) \longmapsto [x_0 y_0, x_0 y_1, \dots, x_m y_0, x_0 y_1, \dots, x_m y_n] = [\overbrace{x_i y_j}^{t_{ij}}]_{\substack{i=0,\dots,m \\ j=0,\dots,n}}$

Claim. φ is an iso onto its image, $\operatorname{Im} \varphi$ is closed.

Well-defined: i.e. $[x_i y_j] \neq [0, \dots, 0] \notin \mathbb{P}^N$

φ is a morphism: cover $\mathbb{P}^N$ by $U_{t_{ij}} = \{t_{ij} \neq 0\}$. $\varphi^{-1}(U_{t_{ij}}) = U_{x_i} \times U_{y_j}$

$\mathcal{O}_{\mathbb{P}^N}(U_{t_{ij}}) = k\left[\frac{t_{k\ell}}{t_{ij}} \,\middle|\, k, \ell\right]$, $\varphi^*\left(\frac{t_{k\ell}}{t_{ij}}\right) = \frac{x_k}{x_i} \cdot \frac{y_\ell}{y_j}$

$\downarrow \varphi^*$ $\qquad\qquad \cap$

$\mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(U_{x_i} \times U_{y_j}) = k\left[\frac{x_0}{x_i}, \dots, \frac{x_m}{x_i}, \frac{y_0}{y_j}, \dots, \frac{y_n}{y_j}\right] \Rightarrow \varphi$ is a morph by Crit.

Injectivity: easy.

Iso: $Z := \varphi(\mathbb{P}^n \times \mathbb{P}^m)$. Suffices to check locally, i.e. $\varphi|_{U_{x_i} \times U_{y_j}} : U_{x_i} \times U_{y_j} \to U_{t_{ij}} \cap Z$ iso

Wlog $i = j = 0$. $\qquad U_{x_0} \times U_{y_0} \longrightarrow U_{0,0} \cap Z$

$$X_i := \frac{x_i}{x_0},\ Y_j := \frac{y_j}{y_0} \qquad T_{ij} := \frac{t_{ij}}{t_{00}}$$

On $U_{00} \cap Z$, $T_{ij} = X_i \cdot Y_j$, $\quad T_{i0} = X_i \underbrace{Y_0}_{1} = X_i$, $\quad T_{0j} = Y_j$

$$\Rightarrow k[U_{00} \cap Z] = k[T_{ij}] / (T_{ij} - T_{i0} T_{0j}) \cong k[T_{10}, \dots, T_{m0}, T_{01}, \dots, T_{0n}]$$

$$\cong \downarrow \varphi^* \qquad \varphi^*(T_{i0}) = X_i,\ \varphi^*(T_{0j}) = Y_j$$

$$k[X_i, Y_j \mid i, j]$$

φ^* iso $\Rightarrow$ φ iso on $U_{00} \cap Z$.

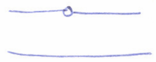
$\mathrm{Im}(\varphi)$ is closed b/c can be covered by fin. many closed, namely $U_{ij} \cap Z$. $\square$

Separatedness

Def. A prevariety is a variety (or separated) if $\forall Y$ prevar $\forall g, h : Y \rightrightarrows X$ morphisms $\mathrm{Eq}(g,h) \subseteq Y$ is closed.

Ex: 1) $Y = \mathbb{A}^1$, $\quad g, h : \mathbb{A}^1 \rightrightarrows X$, $\quad g|_{\mathbb{A}^1 \setminus \{0\}} = h|_{\mathbb{A}^1 \setminus \{0\}}$

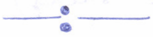
Then $\mathrm{Eq}(g,h) = \begin{cases} \mathbb{A}^1 \setminus \{0\} & \text{if } g(0) \neq h(0) \quad \text{not closed} \\ \mathbb{A}^1 & \text{if } g(0) = h(0) \quad \text{closed} \end{cases}$



So if X vty $\Rightarrow g(0) = h(0)$, "the limit exists and is unique"

More generally: $g : Y \setminus * \longrightarrow X$ & X vty $\Rightarrow$ at most 1 way to extend g to Y.

Converse later.

2) $X =$  , $X_1 = \mathbb{A}^1$, $X_2 = \mathbb{A}^1$, $U_1 = X_1 \setminus \{0\}$, $U_2 = X_2 \setminus \{0\}$, $\varphi : \underset{x}{U_1} \xrightarrow{\cong} \underset{x}{U_2}$

$X = (X_1 \amalg X_2)/\varphi$ $\quad$ Basis of top: $\{(\text{open in } X_1) \cup (\text{open in } X_2)\}$, def str. sh. on basis.

can be extended 2 ways to $\mathbb{A}^1 \longrightarrow X$

3) Every aff var is a var.

$g, h : Y \longrightarrow X$, $\quad X \subseteq \mathbb{A}^n$ irr'ble alg set

$$\mathrm{Eq}(g,h) = \bigcap_{s \in k[X]} \Big(\underbrace{g^*(s) - h^*(s)}_{\substack{\text{regular on } Y \\ \Rightarrow \text{continuous}}}\Big)^{-1}(0) \qquad \text{closed.}$$

$\|$

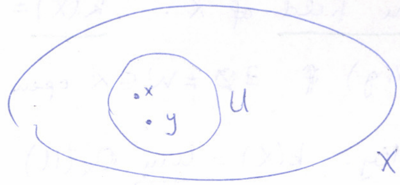
$\{y \in Y \mid g(y) = h(y)\}$

$\Updownarrow$

$\forall s : s(g(y)) = s(h(y))$

Alternatively, $\mathrm{Eq}(g,h) = \mathrm{Ker}(g - h) = \underbrace{(g-h)^{-1}}_{\text{cont.}} \underbrace{(\{0\})}_{\text{closed}}$ $\quad$ closed

4) X_1, X_2 varieties $\Rightarrow$ $X_1 \times X_2$ variety



$g = (g_1, g_2),\ h = (h_1, h_2) : Y \longrightarrow X_1 \times X_2$

$Eq(g, h) = Eq(g_1, h_1) \cap Eq(g_2, h_2)$

5)

Prop. X prevar., $\forall x, y \in X\ \exists U \subseteq X$ open affine, $x, y \in U$ $\Rightarrow$ X variety.

Pf: $g, h : Y \longrightarrow X$, $Z := Eq(g, h)$. Let $z \in \overline{Z}$. Wts $z \in Z$. (Then $Z = \overline{Z}$.)

$x := g(z)$, $y := h(z)$. Let $U \ni x, y$ open affine.

$W := g^{-1}(U) \cap h^{-1}(U) \xrightarrow{g|_W,\ h|_W} U$

By Ex. 4): $\underline{Eq(g|_W, h|_W)}$ is closed. $Eq(g|_W, h|_W) = Eq(g, h) \cap W$

$\underbrace{z \in}\ Eq(g, h) \cap \overline{W} = Eq(g|_W, h|_W) = \overline{Eq(g|_W, h|_W)} = Eq(g, h) \cap W = \underbrace{Z \cap W} \Rightarrow z \in Z$ ✓ □

6) Projective var $\Rightarrow$ var.

Pf: $X \subseteq \mathbb{P}^n$ $\quad n = 0$ ✓ $\quad$ Assume $n \geq 1$.

if $x, y \in X \rightarrow \exists h$ poly, $x, y \in D(h) \in k^n$ affine. We are done by 5) □

7) $X \times X \underset{p_2}{\overset{p_1}{\rightrightarrows}} X$, p_i are the projections

$Eq(p_1, p_2) = \underbrace{\{(x, x) \mid x \in X\}}_{\Delta(X)} \subseteq X \times X$ diagonal

X vty $\Rightarrow$ $\Delta(X)$ closed.

Prop. X prevar. Then X is a vty iff $\Delta(X) \subseteq X \times X$ closed.

Pf: $\Leftarrow$: $g, h : Y \to X$. Consider $g \times h : Y \longrightarrow X \times X$

$Eq(g, h) = (\underbrace{g \times h}_{\text{cont}})^{-1}(\underbrace{\Delta(X)}_{\text{closed}}) \Rightarrow Eq(g, h)$ closed.

Rmk. X top space T2 iff $\Delta(X) \subseteq X \times X$ closed. Note that $X \times X$ here means topological product, not product of prevarieties.

Function field

07.11.2012

X variety

Lemma. $f, g : X \to k$ regular, $\exists \emptyset \neq U \subseteq X$ open. $f|_U = g|_U \Rightarrow f = g$.

Pf: f, g cont since reg. $\Rightarrow$ $\underset{\text{open}}{U} \subseteq \underset{\text{closed in } X}{Eq(f, g)}$

X irr'ble $\Rightarrow$ U dense $\Rightarrow$ $X = Eq(f, g)$

Def. Function field of X: $k(X) = \{(U,f) \mid \emptyset \neq U \subseteq X \text{ open}, \ f: U \to k \text{ regular}\}/\sim$,

$(U,f) \sim (V,g)$ iff $\exists \emptyset \neq W \subseteq U \cap V$ open, $f|_W = g|_W$

Equivalently: $k(X) = \varinjlim_{U \in I} \mathcal{O}_X(U)$ where $I = \{\emptyset \neq U \subseteq X \text{ open}\}$, $U \succeq V :\Leftrightarrow U \subseteq V$

inductive system with restriction maps

Cor. $\mathcal{O}_X(U) \to k(X)$, $f \mapsto [(U,f)]$ is injective

Pf: $f,g \in \mathcal{O}_X(U)$, $[(U,f)] = [(U,g)] \Rightarrow \exists \emptyset \neq W \subseteq U$ open, $f|_W = g|_W \Rightarrow f = g$ by Lemma. □

Rmk. $k(X)$ is a field.

Addition: $(U,f) + (V,g) := (U \cap V, f|_{U\cap V}, g|_{U \cap V})$, multiplication similarly.

Mult. inverse: if $f \neq 0$ then $(U \cap D(f), (f|_{D(f)})^{-1})$ is (an) inverse to (U,f)

Rmk $\mathcal{O}_{X,x} = \varinjlim_{x \in U} \mathcal{O}_X(U)$, $k(X) = \varinjlim_{U} \mathcal{O}_X(U)$, $\mathcal{O}_{X,x} \hookrightarrow k(X)$ injective by Cor.,

$\Rightarrow \mathcal{O}_X(U) = \bigcap_{x \in U} \mathcal{O}_{X,x}$ where we take $\cap$ in $k(X)$. (This is Mumford's def. of $\mathcal{O}_X(U)$)

Rmk. X aff var. $\Rightarrow \mathcal{O}_X(X) = k[X] = R$

$\mathcal{O}_X(D(f)) = R_f$ for $f \neq 0$

$\{D(f) \mid f \neq 0\}$ basis of top. $\Rightarrow$ $k(X) = \varinjlim_{0 \neq f \in k[X]} \mathcal{O}_X(D(f)) = \varinjlim_{0 \neq f \in k[X]} R_f = \bigcup_f \left\{\frac{a}{f^i} \,\middle|\, a \in R\right\}/\sim$,

$\frac{a}{f^i} \sim \frac{b}{g^j} \Leftrightarrow \exists h \in R: \frac{a}{f^i_h} = \frac{b}{g^j_h}$ in $R_h = \mathcal{O}_X(D(h))$

$D(h) \subset D(f)$, $D(h) \subset D(g)$ $\Leftrightarrow \exists m: h^m(ag^j - bf^i) = 0$ in R

Rmk. X vty, $\emptyset \neq U \subseteq X$ open $\Rightarrow k(X) \cong k(U)$

In particular if U is afine then $\underbrace{k(X)}_{\substack{\text{fin gen field} \\ \text{extn of } k}} \cong \operatorname{Frac} R$, $R = \mathcal{O}_X(U)$ is a fin gen k-algebra

Lemma. X (pre)vty, Y vty, $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, $\emptyset \subsetneq U \subseteq X$ open, $f|_U = g|_U$ $\Rightarrow f = g$.

Pf: $U \subseteq \underbrace{\mathrm{Eq}(f,g)}_{\text{closed}} \subseteq X$ □

Def. RatMap$(X,Y) = \{(U,f) \mid \emptyset \neq U \subseteq X, \ f: U \to Y \text{ morphism}\}/\sim$,

$(U,f) \sim (V,g) \Leftrightarrow \exists W \subseteq U \cap V$, $f|_W = g|_W$

Def $[U,f] \in$ RatMap(X,Y) is a rational map from X to Y, denoted $X \dashrightarrow Y$.

f is dominant if $f(U) \subseteq Y$ is dense for some U (or equivalently, for every domain of defn $U \subseteq X$).

($V \subseteq U \subseteq X$ opens, nonempty, $f(U) \subseteq Y$ dense. Then $V \subseteq U$ dense $\Rightarrow f(V) \subseteq Y$ dense.)

Rmk. $f: X \dashrightarrow Y$ dominant rat map, $g: Y \dashrightarrow Z$ any rat map.

$f: U \to Y$, $g: V \to Z$. f dom. $\Rightarrow f(U) \cap V \neq \emptyset \Rightarrow f^{-1}(V) \neq \emptyset$

Define the composition $g \circ f: X \dashrightarrow Z$ by $(f^{-1}(V), g \circ f)$

Def. $f: X \dashrightarrow Y$ rat map is birational if dominant and $\exists g: Y \dashrightarrow X$ dominant s.t. $f \circ g \sim \mathrm{id}_Y$, $g \circ f \sim \mathrm{id}_X$ in $\mathrm{RatMap}(Y,Y)$ resp. $\mathrm{RatMap}(X,X)$.

We say X, Y are birational if $\exists X \dashrightarrow Y$ birat map.

Prop. (exercise) X, Y birational iff $k(X) \simeq k(Y)$.

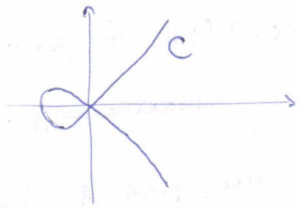
Prop. Contravariant equivalence between

- cat of vties and dom rat maps
- cat of fin gen field extn $/k$.

Ex. $\mathbb{P}^1 \not\simeq \mathbb{A}^1$ but birational.

Rmk. X, Y birat iff $\exists\, \emptyset \neq U \subseteq X,\ \emptyset \neq V \subseteq Y: (U, \mathcal{O}_X|_U) \simeq (V, \mathcal{O}_Y|_V)$

$C: y^2 = x^2(x+1)$, C birat to $\mathbb{P}^1$
(C is rational to $\mathbb{P}^1$)



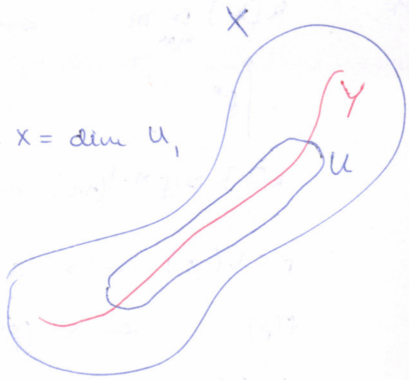
Ex. $C: y^2 = (x+1)x(x-1)$ not rational to $\mathbb{P}^1$ (later)

Dimension

Def. X vty ($\Rightarrow k(X)$ fin gen $/k$). $\dim X := \mathrm{trdeg}_k\, k(X)$

Rmk. $\dim X = n \Leftrightarrow \exists s_1, \dots, s_n \in k(X)$ alg indep. s.t. $k(X)$ is alg ext over $k(s_1, \dots, s_n)$

Prop. $Y \subsetneq X$ proper closed subvty $\Rightarrow \dim X < \dim Y$.



Pf: Let $U \subseteq X$ aff s.t. $U \cap Y \neq \emptyset$. $\Rightarrow k(X) \simeq k(U) \Rightarrow \dim X = \dim U$,
$\dim Y = \dim(U \cap Y)$. Hence wma X, Y affine.

$Y = V(\mathfrak{p})$ for some $\mathfrak{p} \in \mathrm{Spec}\, k[X] \Rightarrow k[Y] = k[X]/\mathfrak{p}$

If $\dim X = \dim Y = n \Rightarrow \exists \bar{s}_1, \dots, \bar{s}_n \in k[X]/\mathfrak{p}$, alg indep.

Lifts: $s_1, \dots, s_n \in k[X]$. Let $y \in \mathfrak{p} \setminus \{0\}$.

$\Rightarrow \exists p(X_1, \dots, X_n, T)$ polynomial with k-coeffs s.t. $p(s_1, \dots, s_n, y) = 0$

$\Rightarrow p(\bar{s}_1, \dots, \bar{s}_n, 0) = 0$ in $k[X]/\mathfrak{p}$.

Since R is an integral domain, wma p to be irr'ble.

Case 1: the constant term in T is non-zero i.e. $p(X_1, \dots, X_n, 0) \neq 0$. $\Rightarrow$ ✓

Case 2. $p = a \cdot T$ where $a \in k^\times$ $\Rightarrow p(y) = ay = 0$ ↯ since $a\, y \neq 0$. $\square$

Def. topdim $X := \sup\{\ell \mid \emptyset \neq Z_0 \subsetneq \dots \subsetneq Z_\ell \subseteq X,\ Z_i$ irred closed$\}$

For a vty X (irred): $\dim Z_{i+1} - \dim Z_i \geq 1$ by Prop. for a chain as above.

$\dim X = \dim Z_\ell \geq \dim Z_{\ell-1} + 1 \geq \dots \geq \dim Z_0 + \ell \geq \ell$

$\Rightarrow \dim X \geq \operatorname{topdim} X$

If $Z_0 \subsetneq \dots \subsetneq Z_\ell = X$ is a maximal chain: can we have $\dim Z_{i+1} - \dim Z_i > 1$?

Thm. (Krull PIT) X vty, $f \in \mathcal{O}_X(X)$, $f \neq 0$, $Z \subseteq V(f)$ irred component $\Rightarrow \dim Z = \dim X - 1$

Cor. $\dim X = \operatorname{topdim} X$

Pf: Suffices to prove: if $Z \subseteq X$ maximal proper irred closed subset then $\dim Z = \dim X - 1$.

To prove this, assume X to be affine. $\Rightarrow Z = V(\mathfrak{a})$, $\mathfrak{a} \subsetneq k[X]$

$\Rightarrow \exists f \in k[X] \setminus \{0\}$ vanishing on Z. Let Z' be an irred component of $V(f)$ containing Z. Maximality $\Rightarrow Z = Z'$ $\Rightarrow \dim Z = \dim X - 1$ by Krull PIT. $\square$

To prepare for the proof of Thm: X, Y aff vty, $f: X \to Y$ morph, $f^*: k[Y] \to k[X]$

Def. f finite $\Leftrightarrow$ $k[X]$ int closed over $f^* k[Y]$

Prop. $f: X \to Y$ finite. Then

1) f is a closed map
2) f has finite fibres
3) f is surjective iff f^* is injective.

Pf:

$$\begin{tikzcd}
k[X] \rhd_{\max} \mathfrak{m} \arrow[r, leftrightarrow] & p \in X \arrow[d, mapsto] \\
k[Y] \rhd (f^*)^{-1}(\mathfrak{m}) \arrow[u, "f^*"] \arrow[r, leftrightarrow] & f(p) \in Y
\end{tikzcd}$$

$p \in V(\mathfrak{a}) \longleftrightarrow \mathfrak{a} \subset \mathfrak{m} \subset k[X]$

$f(p) \longleftrightarrow (f^*)^{-1}(\mathfrak{m})$, $\mathfrak{a} \subseteq \mathfrak{m} \Leftrightarrow (f^*)^{-1}(\mathfrak{a}) \subseteq (f^*)^{-1}(\mathfrak{m})$

$\Rightarrow f(V(\mathfrak{a})) \subseteq V((f^*)^{-1}(\mathfrak{a}))$

Conversely, let $\tilde{\mathfrak{m}} \subseteq k[Y]$ be maximal containing $\mathfrak{a}' = (f^*)^{-1}(\mathfrak{a})$.

$k[Y]/\mathfrak{a}' \hookrightarrow k[X]/\mathfrak{a}$ Going up thm. $\Rightarrow \tilde{\mathfrak{m}}/\mathfrak{a}' = \mathfrak{m}' \cap k[Y]/\mathfrak{a}$

for some $\mathfrak{m}' \subset k[X]/\mathfrak{a}$. $\Rightarrow \tilde{\mathfrak{m}} = (f^*)^{-1}(\mathfrak{m})$, $\mathfrak{m} = \mathfrak{m}' + \mathfrak{a} \subseteq k[X]$ maximal

$\Rightarrow \supseteq \Rightarrow f(V(\mathfrak{a}))$ closed $\Rightarrow$ 1).

Rmk. A ring $\Rightarrow$ Krull dim $A = \{\sup \ell \mid p_0 \subsetneq \cdots \subsetneq p_\ell \subseteq A\}$ prime ideals

For X affine: top dim X = Krull dim $k[X]$

Def. X, Y affine, $f: X \to Y$ is finite if $k[X]$ int dependent on $f^*k[Y]$.

Prop. If $f: X \to Y$ is finite then

1) f is closed
2) f has finite fibres
3) f surj $\Leftrightarrow$ f^* inj
4) f surj $\Rightarrow$ $\dim X = \dim Y$

Pf: 1) $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$

$$\begin{array}{ccccccc} k^n \supseteq X \supseteq V(\mathfrak{a}) \ni p & \longleftrightarrow & \mathfrak{m} \supseteq \mathfrak{a} \text{ max ideal} \\ \downarrow f \qquad\qquad \downarrow & & \\ Y \qquad\qquad \ni f(p) & \longleftrightarrow & (f^*)^{-1}(\mathfrak{m}) \supseteq (f^*)^{-1}(\mathfrak{a}) \end{array}$$

$$\begin{array}{cc} k[X] \supseteq & \mathfrak{a} \\ f^* \uparrow & \uparrow \\ k[Y] & (f^*)^{-1}(\mathfrak{a}) \end{array}$$

$f(V(\mathfrak{a})) = V((f^*)^{-1}(\mathfrak{a}))$

$\supseteq$: f finite by going up

$\subseteq$

2) Let $Q \in Y$ with corresponding max ideal $\tilde{\mathfrak{m}} \subseteq k[Y]$.

Points $P \in f^{-1}(Q)$ corresp to max ideals $\mathfrak{m} \subseteq k[X]$ s.t. $(f^*)^{-1}(\mathfrak{m}) = \tilde{\mathfrak{m}}$

$\Rightarrow f^*(\tilde{\mathfrak{m}}) \subseteq \mathfrak{m}$

Thus points in $f^{-1}(Q)$ correspond to max ideals in $\underbrace{k[X] / f^*(\tilde{\mathfrak{m}}) \cdot k[X]}$
int. dep. on $k[Y]/\tilde{\mathfrak{m}} = k$

$\Rightarrow k[X]/f^*(\tilde{\mathfrak{m}})k[X]$ is fin dim over k

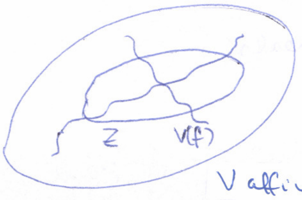
$\Rightarrow$ fin many max ideals. ✓

3) f surj $\Rightarrow$ f^* inj ✓

Conversely: $X = V(0)$. By pf of 1): $f(X) = V((f^*)^{-1}(0)) = V(0) = Y$.

4) $$\begin{array}{ccc} k[X] & & k(X) \\ \uparrow \text{int dep} & & \uparrow \text{alg dep} \\ k[Y] & & k(Y) \end{array} \Rightarrow \text{trdeg } k(X) = \text{trdeg } k(Y).$$

Pf of PIT: $\underset{\text{Step 1.}}{\dim}\ X = \dim V, \quad \dim Z = \dim Z \cap V$



V affine

$\Rightarrow$ wma X is affine.

Step 2. Reduce to the case $V(f) = Z$.

Let $V(f) = \underbrace{Z_1}_{Z} \cup \dots \cup Z_n$, let $f_i \in k[X]$ vanish on Z_i but not on Z_1, $i \geq 2$

Restrict to $D(f_2 \cdots f_n) = Z \cap D(f_2 \cdots f_n) = V(f) \cap D(f_2 \cdots f_n) = V(f|_{D(f_2 \cdots f_n)})$

Replace X by $D(f_2 \cdots f_n)$ and f by $f|_{D(f_2 \cdots f_n)}$

Step 3. Reduce to the case $X = \mathbb{A}^n$.

NN $\Rightarrow \exists x_1, \dots, x_n \in k[X]$, $k[X]$ is int dep on $k[x_1, \dots, x_n]$

$\Leftrightarrow \exists\ X \supset Z = V(\wp) \qquad \wp = \sqrt{(f)}$

$$\begin{array}{ccc} X & \supset & Z = V(\wp) \\ {\scriptstyle \text{finite}}\ \downarrow \pi & & \downarrow \\ \mathbb{A}^n & & \pi(Z) = V((\pi^*)^{-1}\wp) = V(\wp \cap k[x_1, \dots, x_n]), \end{array} \qquad \dim X = \dim \mathbb{A}^n$$

$\pi|_Z$ finite $\Rightarrow \dim Z = \dim \pi(Z)$. Nts $\pi(Z)$ is int.

Idea:
$$\begin{array}{ccc} k(X) & \longleftrightarrow & k[X] \ni f \\ \uparrow & & \uparrow \pi^* \\ K = k(x_1, \dots, x_n) & \longleftrightarrow & k[x_1, \dots, x_n] \end{array}$$

Let $f_0 := \mathrm{Norm}_{k(X)/K}(f)$

Claim 1. $f_0 \in k[x_1, \dots, x_n]$

Claim 2. $\sqrt{(f_0)} = \underbrace{\wp \cap k[x_1, \dots, x_n]}_{I(\pi(Z))}$.

Pf of Cl 1: Let $p(Y) := Y^n + \dots + a_n \in K[Y]$ be the min poly of f

Rmk. $m_f : k(X) \longrightarrow k(X)$, $y \longmapsto fy$ K-lin map, $\mathrm{Norm}_{k(X)/K}(m_f) = \det m_f \in K$

Then p is also the min. poly. of m_f.

Roots of p are the conjugates of $f \in k(X)/K$: $\quad f_1, \dots, f_\ell$

f is int. dependent on $k[x_1, \dots, x_n] \Rightarrow \exists\ \tilde{p}$ monic, $\tilde{p} \in k[x_1, \dots, x_n][Y]$, $\tilde{p}(f) = 0$

$\Rightarrow \tilde{p}(f_i) = 0\ \forall i. \qquad \left(p \mid \tilde{p},\ \tilde{p} = p \cdot q,\ \tilde{p}(f_i) = \underbrace{p(f_i)}_{0} \cdot q(f_i)\right)$

$\Rightarrow f_i$ int dep on $k[x_1, \dots, x_n] \Rightarrow a_i$ int dependent.

But $a_i \in K$ and $k[x_1, \dots, x_n]$ int closed in $K \Rightarrow a_i \in k[x_1, \dots, x_n]$

$\underset{\ell \geq 1}{\mathrm{CharPol}(m_f) = (\mathrm{MinPol}(m_f))^\ell} \Rightarrow f_0 = a_n^\ell \in k[x_1, \dots, x_n]$. $\boxed{1}$

Pf of Cl 2: $\rho(f)=0 \Rightarrow \underbrace{f^n+\dots+a_{n-1}f}_{\in (f)}+a_n=0 \Rightarrow a_n\in (f)\cap k[x_1,\dots,x_n]$

$\Rightarrow f_0=a_n^l \in \sqrt{(f)}\cap k[x_1,\dots,x_n]=\wp\cap k[x_1,\dots,x_n] \Rightarrow \subseteq$

Conversely, let $g\in \wp\cap k[x_1,\dots,x_n]$.

$\Rightarrow \exists m\geq 1: \quad g^m\in (f) \Rightarrow \exists h\in k[X]: \ g^m=h\cdot f$

$\underbrace{\mathrm{Norm}(g)^m}_{g^{m[k(X):K]}}=\underbrace{\mathrm{Norm}(h)}_{\in k[x_1,\dots,x_n]}\ \underbrace{\mathrm{Norm}(f)}_{f_0} \Rightarrow g^{m[k(X):K]}=f_0\cdot \underset{\in k[x_1,\dots,x_n]}{h_0} \Rightarrow g\in\sqrt{(f_0)}$

Step 4. $X=\mathbb{A}^n$, $Z=V(f)$ irr'ble, $\wp=I(Z)=\sqrt{(f)}$

$k[x_1,\dots,x_n]$ UFD, Z irr'ble $\Rightarrow f=u\cdot g^l$, u unit, $g\in k[x_1,\dots,x_n]$ irr'ble

$\operatorname{trdeg} k(Z)=\operatorname{trdeg}\operatorname{Frac}\left(k[x_1,\dots,x_n]/(g)\right)=n-1$

Cor. X vty, $f_1,\dots,f_r\in \mathcal{O}_X(X)$. Then every irred component of $V(f_1,\dots,f_r)$ has codim $\leq r$. (Recall: $\operatorname{codim}(Z,X):=\dim X-\dim Z$)

Pf: $Z\subset V(f_1,\dots,f_r)$ irred comp, Z' component of $V(f_1,\dots,f_{r-1})$ containing Z

Induction $\Rightarrow \operatorname{codim}(Z',X)\leq r-1$.

$Z\subseteq Z'\cap V(f_r)\subseteq V(f_1,\dots,f_r)$, Z is an irred comp of $V(f_1,\dots,f_r)$, thus also irred comp of $Z'\cap V(f_r)=V(f_r|_{Z'})$

(Z closed in X, X closed in Y, $Z\subseteq Y$ irred $\Rightarrow Z\subseteq X$ irred holds in general)

- If $f_r|_{Z'}=0 \Rightarrow Z=Z' \Rightarrow \operatorname{codim} Z=\operatorname{codim} Z'$
- If $f_r|_{Z'}\neq 0 \overset{\text{PIT}}{\Rightarrow} \operatorname{codim} Z=\underbrace{\operatorname{codim} Z'}_{\leq r-1}+1 \Big\}\leq r$

Prop. (Converse) X aff, $Z\subseteq X$ irred closed, $\operatorname{codim}(Z,X)=r \Rightarrow \exists f_1,\dots,f_r\in k[X]$, Z component of $V(f_1,\dots,f_r)$ and every component of $V(f_1,\dots,f_r)$ has codim r.

Pf: Let $Z=Z_r\subsetneq Z_{r-1}\subsetneq\dots\subsetneq Z_1\subsetneq X$ be a chain of irred closed subsets.

Claim. $\exists f_1,\dots,f_r\in k[X]$ s.t. 1) Z_i irred comp of $V(f_1,\dots,f_i)$ and 2) every irred comp of $V(f_1,\dots,f_i)$ has codim i.

Induction: $i=1$. $0\neq f_1\in I(Z_1)$, let Z' be an irred comp of $V(f_1)$ containing Z

$\underbrace{Z_1}_{\text{codim }1}\subseteq \underbrace{Z'}_{\substack{\text{codim}=1\\ \text{by PIT}}}\subsetneq X \Rightarrow Z_1=Z'$ like before.

Induction step. $Y_1,\dots,Y_l$ components of $V(f_1,\dots,f_{i-1})$, $Y_1=Z_{i-1}\supsetneq Z_i$

Since $Y_j\setminus Z_i\neq\emptyset$ $(j\geq 1)$: $\exists g_j\in I(Z_i)$ not vanishing on Y_j, $j=1,\dots,l$

$\Rightarrow I(Z_i) \not\subseteq I(Y_j)$, $j = 1, \dots, \ell$, $I(Y_j)$ is prime

Prime Avoidance $\Rightarrow I(Z_i) \not\subseteq \bigcup_j I(Y_j)$. Let $f_i \in I(Z_i) \setminus \bigcup_j I(Y_j)$. Check that this f_i does the job, (similar to $i=1$). □

Exc./Warning. Given $Z \subseteq X$ we may not always find $f_1, \dots, f_r \in k[X]$ s.t. $V(f_1, \dots, f_r) = Z$, even if $r = 1$.

Ex. $S = (y^2 z = (x+2)x(x-2)) \subseteq \mathbb{A}^3$, $L \subseteq S$ line through the origin. If L is chosen correctly then $\nexists f \in k[S]$ for which $V(f) = L$, even set theoretically.

Prop: $X, Y \subseteq \mathbb{A}^n$, $\operatorname{codim}(X, \mathbb{A}^n) = e$, $\operatorname{codim}(Y, \mathbb{A}^n) = f$.
Then every component of $X \cap Y$ has codimension $\leq e + f$.

Pf: $X \cap Y \simeq (X \times Y) \cap \Delta_{k^n}$ where $\Delta_{k^n} = V(x_1 - y_1, \dots, x_n - y_n)$

$= V\left((x_i - y_i)|_{X \times Y} \mid i = 1, \dots, n \right)$ is of codim $\leq n$ in $X \times Y$

$n \geq \operatorname{codim}(X \cap Y, X \times Y) = \dim X + \dim Y - \dim(X \cap Y)$

$= n - e + n - f - \dim(X \cap Y)$

$\Rightarrow \dim(X \cap Y) \geq n - e - f$. □

Prop. $X, Y \subseteq \mathbb{P}^n$ proj varieties of codim e and f. Then

1) Every component of $X \cap Y$ has codim $\leq e + f$.

2) If $e + f \leq n$ then $X \cap Y \neq \emptyset$.

Pf: $X = V(\mathfrak{a})$, $\mathfrak{a} \subseteq k[x_0, \dots, x_n]$ homogeneous

$X^* := V_{k^{n+1}}(\mathfrak{a}) \subseteq k^{n+1}$

$X^* \cap D(x_i) \xrightarrow{\sim} (X \cap U_i) \times (\mathbb{A}^1 \setminus \{0\})$

$(x_0, \dots, x_n) \longmapsto ([x_0, \dots, x_n], x_i)$

$\left(t \frac{a_0}{a_i}, \dots, t \frac{a_n}{a_i}\right) \longmapsfrom ([a_0, \dots, a_n], t)$

$\rightarrow \dim X^* \underset{\exists i}{=} \dim \left(X^* \cap D(x_i)\right) = \dim(X \times \mathbb{A}^1 \setminus \{0\}) = \dim X + \dim \mathbb{A}^1$

$= \dim X + 1$

$\Rightarrow \operatorname{codim}(X, \mathbb{P}^n) = e = \operatorname{codim}(X^*, k^{n+1})$

Let $e + f \leq n$. Since $0 \in X^* \cap Y^*$, $\exists$ irred. comp. $Z \subseteq X^* \cap Y^*$

Prop. $\Rightarrow \operatorname{codim}(Z, k^{n+1}) \geq e + f$

$\Rightarrow \dim Z \geq n + 1 - e - f \geq 1$

$\Rightarrow \exists\, 0 \neq (a_0, \dots, a_n) \in Z$

$\Rightarrow [a_0, \dots, a_n] \in X \cap Y$. □

<u>Rmk.</u> X aff var. Then $\dim X = n \iff \exists$ finite surjective morphism $X \twoheadrightarrow \mathbb{A}^n$

Pf: NNT $\Rightarrow \exists f: X \twoheadrightarrow \mathbb{A}^n$ fin surj.

Nts $n = \dim X$.

$$\begin{array}{ccc} k[X] & \hookrightarrow & \underbrace{k[X] \otimes_{k[x_1,\dots,x_n]} K}_{k(X)} \\ \text{int. dependent} \uparrow \varphi & & \uparrow \text{int dependent} \\ k[x_1,\dots,x_n] & \hookrightarrow & k(x_1,\dots,x_n) = K \end{array}$$

localise the fuctions pulled back from $\mathbb{A}^n$

Since $k[X] \otimes K$ int domain and finite K-module, int dependent

$\Rightarrow \underbrace{k[X] \otimes K}_{k(X)}$ field $\Rightarrow k(X)$ int dep over K $\Rightarrow$ algebraic

$\Rightarrow \operatorname{trdeg} k(X) = \operatorname{trdeg} K = n.$

Fibres of morphisms

$$\begin{array}{ccc} X & \supseteq & f^{-1}(\{y\}) \\ f \downarrow & & \downarrow \\ Y & \ni & y \end{array}$$

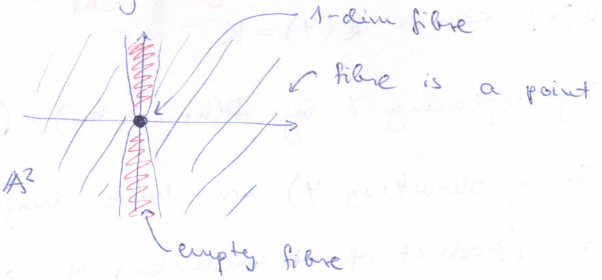
How does $\dim(f^{-1}(y))$ vary with y?

<u>Ex.</u> $f^{-1}(y) = \begin{cases} pt & y \in U \\ \emptyset & y \notin U \end{cases}$ if $U \subseteq Y$ open, $f: U \hookrightarrow Y$ inclusion

<u>Ex.</u> $f: \mathbb{A}^2 \longrightarrow \mathbb{A}^2$, $(x,y) \longmapsto (x, xy)$. $f^{-1}(a,b) = \{(x,y) \mid x = a, xy = b\}$

If $a \neq 0$: $x = a$, $y = \frac{b}{a}$. If $a = 0$: $x = 0$, y arbitray, $b = 0$

$\Rightarrow f^{-1}(a,b) = \begin{cases} \{(a, \frac{b}{a})\} & a \neq 0 \\ 0 \times k^1 & a = b = 0 \\ \emptyset & a = 0,\ b \neq 0 \end{cases}$



Now let $f: X \to Y$ be a morphism,

and wma $f(X) \subseteq Y$ to be dense. (If not, replace Y by $\overline{f(X)}$ with the induced vty structure)

<u>Rmk.</u> X vty, $Z \subseteq X$ closed irr'ble. Define $\mathcal{O}_Z$ by

$\mathcal{O}_Z(U) = \{ f: U \to k \mid \exists \tilde{U} \subseteq X \text{ open}, \tilde{U} \cap Z = U, \exists \tilde{f} \in \mathcal{O}_X(\tilde{U}): \tilde{f}|_U = f \} \Rightarrow (Z, \mathcal{O}_Z)$ variety

<u>Prop.</u> Let $r = \dim X - \dim Y$, $y \in Y$. Then every irreducible component of $f^{-1}(y)$ has dimension $\geq r$.

Pf: Wma Y to be affine, $\dim Y = n$.

By a Prop from lecture 8: $\exists g_1, \dots, g_n \in k[Y]$: $V(g_1, \dots, g_n) = \{\underset{\overset{\shortparallel}{y}}{y_1}, \dots, y_n\}$.

Let $\tilde{g}_i := f^*(g_i)$, $V(\tilde{g}_1, \dots, \tilde{g}_n) = \underset{\underset{Z \text{ irred.}}{\cup\!\!\!\!\cup}}{f^{-1}(y)} \amalg f^{-1}(y_2) \amalg \dots \amalg f^{-1}(y_n)$

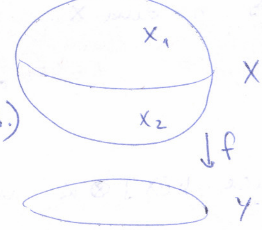
<u>Prop. 2.</u> $Z \subseteq V(\tilde{g}_1, \dots, \tilde{g}_n)$ irred comp $\Rightarrow \operatorname{codim}(Z, X) \leq n - \dim Y$. $\Rightarrow \dim X - \dim Z \leq n - \dim Y$.

Thm. $f: X \to Y$ dominant, $r = \dim X - \dim Y$, $\exists U \subseteq Y$ open s.t.

1) $U \subseteq f(X)$, 2) $\forall y \in U$ $\forall Z \subseteq f^{-1}(y)$ irred comp.: $\dim Z = r$

Pf: Assume Y to be affine (as before). We may also assume X to be affine.

(If X not affine: let $X = \bigcup_i X_i$ affine cover, assume the Thm. holds for $f|_{X_i}$ $\forall i$, pick U_i's as in Thm. $\Rightarrow U := \bigcap U_i$ does the job.)



Idea. Use NNT relative to Y.

Claim. $\exists \emptyset \neq U \subseteq Y$ open, π finite surjective s.t. this commutes:

$$\begin{tikzcd} X \arrow[d, "f"'] & f^{-1}(U) \arrow[l, hook'] \arrow[d, "\pi"] \\ & U \times \mathbb{A}^r \arrow[d, "pr"] \\ Y & U \arrow[l, hook'] \end{tikzcd}$$

Claim $\Rightarrow$ Thm.: 1) holds because π is surjective.

2): Let $y \in U$. $f^{-1}(y) = \pi^{-1}(y \times \mathbb{A}^r) \supseteq Z$ irred component.

$$\dim Z \underset{\pi \text{ finite}}{=} \dim(\pi(Z)) \underset{\pi(Z) \subseteq y \times \mathbb{A}^r}{\leq} \dim(y \times \mathbb{A}^r) = r$$

We have $\geq$ too by Prop. This proves the Thm.

PF OF CLAIM:

$$\begin{tikzcd} t \in k[X] \arrow[r, hook] & k[X] \otimes_{k[Y]} K \ (\subseteq k(X)) \\ & K[Y_1, \dots, Y_n] \arrow[u, "\text{int dependent}"'] \\ k[Y] \arrow[uu, "f^*"] \arrow[r, hook] & k(Y) = K \arrow[uu] \arrow[u, "NN"'] \end{tikzcd}$$

$\uparrow$ int dep.?? $k[Y][Y_1, \dots, Y_n]$

Apply NN to $k[X] \otimes K$ (rel K) $\Rightarrow \exists Y_1, \dots, Y_n$ as in the diagram

$$Y_i = \sum_j \underbrace{g_{ij}}_{\in k[X]} \otimes \underbrace{h_{ij}}_{\in K} = \frac{g_i}{h_i}, \quad g_i \in k[X], \ h_i \in k[Y]$$

By replacing Y by $D(h_1, \dots, h_n) = \bigcap D(h_i)$ wma $h_i = 1 \Rightarrow Y_i \in k[X]$.

Let $t \in k[X] \Rightarrow \exists a_i \in K[Y_1, \dots, Y_n]$: $t^m + a_1 t^{m-1} + \dots + a_m = 0$, $a_i = \sum_I \frac{c_I}{d_I} Y^I$

$b :=$ (product of denominators of coeffs of a_i) $= \prod_I d_I \in k[Y]$

$a_i \in k[Y]_b[Y_1, \dots, Y_n] \Rightarrow t$ int dep

Now let $t_1, \dots, t_s$ be generators of $k[X]$, $b_1, \dots, b_s$ be as above

$\Rightarrow t_1, \dots, t_s$ int dep on $k[X]_b$, $k[Y]_{b_1 \dots b_s}[Y_1 \dots Y_n]$

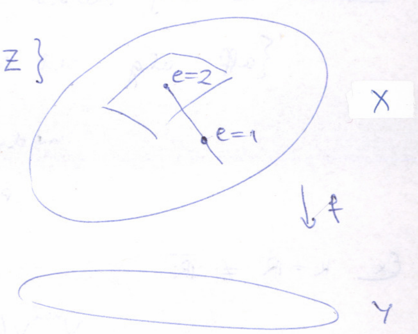
Set $U := D(b_1 \cdots b_s) \Rightarrow$

$$\begin{array}{ccc} k[X]_{b_1 \dots b_s} & & f^{-1}(U) \\ \uparrow \text{int dep} & & \downarrow \text{fin} \\ k[Y]_{b_1 \dots b_s} & & U \times \mathbb{A}^r \\ \uparrow & & \downarrow \\ k[Y]_{b_1 \dots b_s} & & D(b_1 \dots b_s) \end{array}$$

$\square$

<u>Cor.</u> $f: X \longrightarrow Y$ morph. of varieties, $x \in X$.

$e(x) := \max\{\dim Z \mid Z \subseteq f^{-1}(f(x)) \text{ irreducible component}, x \in Z\}$



a) e is upper semicontinuous, i.e. $e^{-1}([n, +\infty))$ is closed for all n

b) If f is closed then

$\tilde{e}(y) := \max\{\dim Z \mid Z \subseteq f^{-1}(y) \text{ irred comp}\}$ is upper semicont.

("Fibre dimensions jump up on closed subsets".)

Pf: a) Induction on $\dim Y$.

$\dim Y = 0$ trivial. Let $r := \dim X - \dim Y$.

If $n \leq r$: $\{x \in X \mid e(x) \geq n\} = X$ by the Prop which stated that fibre dim $\geq r$

If $n > r$: $\{x \in X \mid e(x) \geq n\} \subseteq f^{-1}(U)^c$

Restrict f to $f^{-1}(U)^c = X_1 \cup \ldots \cup X_\ell$ irred comp.

$U^c = Y_1 \cup \ldots \cup Y_\ell$ possibly with repetition $(Y_i := \overline{f(X_i)})$

$f_i := f|_{X_i} : X_i \longrightarrow Y_i$

$\{x \in X \mid e_f(x) \geq n\} = \bigcup \underbrace{\{x \in X_i \mid e_{f_i}(x) \geq n\}}_{\text{closed by induction}}$ This proves a)

b) f closed. $\Rightarrow \{y \in Y \mid \tilde{e}(y) \geq n\} = f(\underbrace{\{x \in X \mid e(x) \geq n\}}_{\text{closed by a)}})$ closed.

<u>Cor.</u> (Chevalley's Thm.) $f: X \longrightarrow Y$ morphism $\Rightarrow f(X)$ is a constructible subset of Y.
More generally, img of constructible is constructible.

<u>Def.</u> $A \subseteq X$ constructible $\Leftrightarrow A = \coprod_{i=1}^{n} A_i$ locally closed, i.e. closed inside an open set.

$\{$constructible subsets of $X\}$ = smallest set of subsets of X containing the open subsets and stable under finite $\cup, \cap, (\cdot)^c$

Pf of Cor: It suffices to show the first assertion, i.e. that $f(X)$ is constructible.
Induction on $\dim Y$. If f is not dominant, replace Y by $\overline{f(X)}$, and apply induction. If f is dominant: $\emptyset \neq U \subseteq f(X)$ open,
$f(X) = U \amalg (U^c \cap f(X))$. Now replace X, Y, f by $f^{-1}(U^c), U^c, f|_{f^{-1}(U^c)}$.

Affine schemes

Recall. k alg closed

$$\{\text{aff alg set in } k^n\} \underset{I}{\overset{V}{\longleftrightarrow}} \{\text{rad ideals in } k[x_1,\dots,x_n]\}$$

irreducible — prime

point — maximal

Ex. $k = \mathbb{R} \neq \overline{\mathbb{R}}$ $\quad \mathbb{R}^2 \rightsquigarrow \mathbb{R}[x,y]$

$V^{\text{naive}}(f) = \emptyset \qquad f = x^2+y^2+1$

$I^{\text{naive}}(V^{\text{naive}}(f)) = \mathbb{R}[x,y] \qquad \to \mathbb{R}^2$ does not have "enough" points

Def. R commutative with unit.

$\mathrm{Spec}(R) := \{\mathfrak{p} \subseteq R \mid \mathfrak{p} \text{ prime ideal}\}$

Def. $\mathfrak{a} \subseteq R$. $V(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Spec}\, R \mid \mathfrak{a} \subseteq \mathfrak{p}\}$

Prop. $C = \{V(\mathfrak{a}) \mid \mathfrak{a} \subseteq R \text{ ideal}\}$ is the collection of closed subsets of a topology on Spec R, called the Zariski topology. $\square$

Def. $f \in R$, $\mathfrak{p} \in \mathrm{Spec}\, R$. The value of f is defined as:

- if $\mathfrak{p}$ is maximal: $f(\mathfrak{p}) :=$ img of f under $R \to R/\mathfrak{p}$
- in general, let $f(\mathfrak{p}) :=$ img of f under $R \to R/\mathfrak{p} \to \mathrm{Frac}(R/\mathfrak{p})$

Def. Residue field: $k(\mathfrak{p}) := \mathrm{Frac}(R/\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$

Lemma. $V((f)) = \{\mathfrak{p} \in \mathrm{Spec}\, R \mid f(\mathfrak{p}) = 0\}$ $\square$

Def. $D(f) := \{\mathfrak{p} \in \mathrm{Spec}\, R \mid f(\mathfrak{p}) \neq 0\}$

Lemma. $\{D(f) \mid f \in R\}$ is a basis of the topology on Spec R. $\square$

Thm. $\{\text{closed subsets of Spec} R\} \underset{V,\,I}{\overset{1:1}{\longleftrightarrow}} \{\text{radical ideals in } R\}$

$$V(\mathfrak{a}) \longleftarrow \mathfrak{a}$$

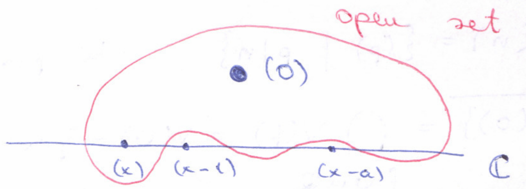
$$X \longmapsto I(X) := \{f \in R \mid \forall \mathfrak{p} \in X : f(\mathfrak{p}) = 0\}$$

Pf: Surjectivity of V: sts $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$

$$I(V(\mathfrak{a})) = \{f \in R \mid f(\mathfrak{p}) = 0 \quad \forall \mathfrak{p} \in V(\mathfrak{a})\}$$
$$= \{f \in R \mid f \in \mathfrak{p} \quad \forall \mathfrak{p} \supseteq \mathfrak{a}\} = \bigcap_{\substack{\mathfrak{p} \supseteq \mathfrak{a} \\ \text{prime}}} \mathfrak{p} = \sqrt{\mathfrak{a}}.$$

$\square$

Note: this would have failed if we worked with mSpec R, because $\bigcap \mathfrak{m} = \mathrm{Jac}(R) \supsetneqq \mathrm{Nil}(R)$ in general.



Ex. 1) $R = \mathbb{C}[x]$

$\operatorname{Spec} R = \{(x-a) \mid a \in \mathbb{C}\} \cup \{(0)\}$

$V(f) = \{\mathfrak{p} \subseteq \mathbb{C}[x] \mid f \in \mathfrak{p}\} = \{(x-a) \mid f(a) = 0\}$
$f \neq 0$

$D(f) = \{(x-a) \mid f(a) \neq 0\} \cup \{(0)\}$

Open: contains everything with the exception of finitely many points.

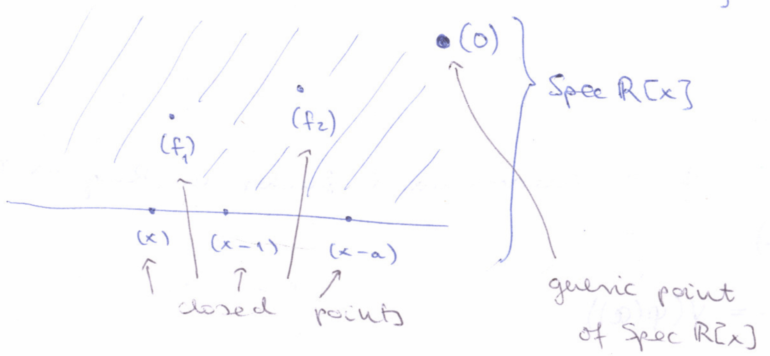
Def. $\mathfrak{p} \in \operatorname{Spec} R$ is closed if $\{\mathfrak{p}\} \subseteq \operatorname{Spec} R$ is closed.

Def. $Z := \overline{\{\mathfrak{p}\}}$ is irreducible. We call $\mathfrak{p}$ a generic point of Z.

Ex. 1) $R = \mathbb{C}[x]$ again. $V(x-a) = \{(x-a)\}$, $(x-a)$ closed pt.

$\overline{\{(0)\}} = \bigcap_{\substack{f \in R \\ (0) \in V(f)}} V(f) = V(0) = \operatorname{Spec} R$ (0) generic pt of $\operatorname{Spec} R$

Ex. 2) $R = \mathbb{R}[x]$, $\operatorname{Spec} \mathbb{R}(x) = \{(x-a) \mid a \in \mathbb{R}\} \cup \{(f) \mid f \in \mathbb{R}[x] \text{ irred. deg } 2\}$



$\{(f)\} = V(f)$

$\varphi : R \to S$ ring hom., $\mathfrak{p} \in S$ prime $\Rightarrow \varphi(\mathfrak{p})$ prime.

Def. $\varphi^* : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$

$\mathfrak{p} \longmapsto \varphi^{-1}(\mathfrak{p})$

Ex. 2) $\varphi : \mathbb{R}[x] \longrightarrow \mathbb{C}[x]$

$\varphi^*((0)) = \varphi^{-1}((0)) = (0)$ Img of generic pt is generic pt.

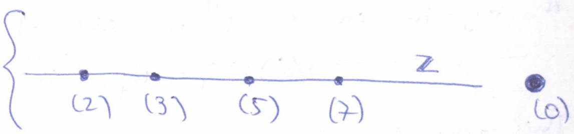
$$\varphi^*((x-a)) = \varphi^{-1}((x-a)) = \begin{cases} (x-a) & a \in \mathbb{R} \\ (x-a)(x-\bar{a}) & a \in \mathbb{C} \setminus \mathbb{R} \end{cases}$$

Ex. 3) $R = \mathbb{Z}$, $\operatorname{Spec} \mathbb{Z} = \{(p) \mid p \text{ prime}\} \cup \{(0)\}$

Functions: $f = n \in \mathbb{Z}$

Values of f: $\mathfrak{p} = (0)$: $f(\mathfrak{p}) = $ img of n in $\operatorname{Frac}(\mathbb{Z}/0) = \mathbb{Q}$

Spec $\mathbb{Z}$



$\mathfrak{p} = (p)$: $f(\mathfrak{p}) = $ img of n in $\operatorname{Frac}(\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$

E.g. $f = 10$, $f((2)) = 0$, $f((3)) = 1$, $f((5)) = 0$, $f((7)) = 3$

$V(10) = \{(2), (5)\}$

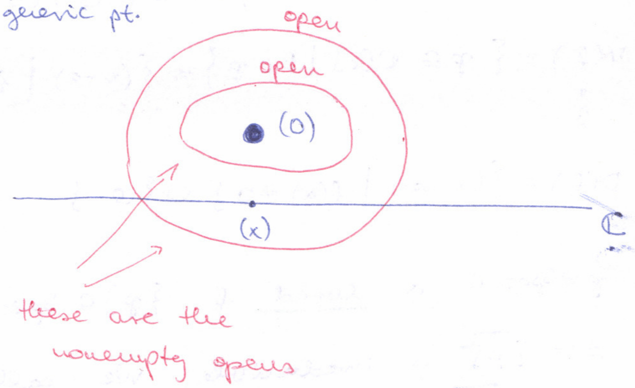
$V(n) = \{(p) \mid p \mid n\}$ In part., $V(p) = \{(p)\}$, i.e. (p) is a closed pt.

$\overline{\{(0)\}} = \bigcap_{f=0 \text{ in } \mathbb{Q}} V(f) = V(0) = \operatorname{Spec} \mathbb{Z}$, (0) generic pt.

<u>Ex.</u> $R = \mathbb{C}[x]_{(x)}$, $\operatorname{Spec} R = \{(0), (x)\}$

$V(x) = \{(x)\}$ is closed

$V((0)) = \operatorname{Spec} R$



<u>Lemma.</u> $\operatorname{Spec} R$ is qc.

19.11.2018

Pf: $\operatorname{Spec} R = \bigcup_i U_i$, pass to a subcover $\longrightarrow$ wma $U_i \subseteq D(f_i)$, $f_i \in R$

$\emptyset = \bigcap_{i \in I} V(f_i) = V\left(\sum_i (f_i)\right) \Rightarrow R = \sqrt{\sum_{i \in I} (f_i)} \ni 1 \Rightarrow 1 = f_1 g_1 + \ldots + f_n g_n$

after reindexing $\Rightarrow X = D(f_1) \cup \ldots \cup D(f_n)$ □

$\varphi: S \longrightarrow R$ in Ring

$\varphi^*: \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$

$\mathfrak{p} \longmapsto \varphi^{-1}(\mathfrak{p})$

<u>Prop.</u> $R \xmapsto{\operatorname{Spec}} \operatorname{Spec} R$ is a contravariant functor Ring $\longrightarrow$ Top

$(\varphi: R \to S) \longmapsto (\varphi^*: \operatorname{Spec} S \longrightarrow \operatorname{Spec} R)$

Pf. Show φ^* is cont: $(\varphi^*)^{-1} V(\mathfrak{a}) = \ldots = V(\varphi(\mathfrak{a}))$ □

<u>Lemma.</u> $f \in R$, $\varphi: R \longrightarrow R_f$ localisation. Then $\varphi^*: \operatorname{Spec} R_f \longrightarrow \operatorname{Spec} R$ is an open embedding with img $D(f)$, i.e. a homeomorphism onto $D(f)$.

Pf: Bijection by std props of loc. ✓ Continuity by Prop. ✓

Remains: openness. $V(\mathfrak{a}) \subseteq \operatorname{Spec} R_f$, $\mathfrak{a} \subseteq R_f$

$\varphi^*(V(\mathfrak{a})) = \underbrace{V(\varphi^{-1}(\mathfrak{a})) \cap D(f)}_{\text{closed in } D(f)}$ + bijective $\Rightarrow \varphi^*$ open. □

<u>Rmk.</u> $I \subseteq R$ ideal, $\varphi: R \to R/I$. Then $\operatorname{Spec}(R/I) \longrightarrow \operatorname{Spec} R$ closed embedding w/ img $V(I)$. □

Structure sheaf

$X = \operatorname{Spec} R$.

A reg fn on X "is" an elt of R,

— " — $D(f) \simeq \operatorname{Spec} R_f$ "is" an elt of R_f.

This already tells us what the str sheaf should be.

$\mathcal{B} := \{D(f) \mid f \in R\}$ basis, $\mathcal{O}_X^{(\mathcal{B})}(D(f)) := R_f$

Restriction maps: $D(f) = D(g) \Leftrightarrow \sqrt{(f)} = \sqrt{(g)} \Leftrightarrow f^m = g \cdot h$ for some $h \in R$

Localisation $R_f \longrightarrow R_g$ is the restriction map.

$$\mathcal{O}^{\mathcal{B}}_{X,\mathfrak{p}} = \varinjlim_{D(f)\ni\mathfrak{p}} \mathcal{O}^{\mathcal{B}}_X(D(f)) = \varinjlim_{f(\mathfrak{p})\neq 0} R_f = (R\setminus\mathfrak{p})^{-1}R = R_{\mathfrak{p}}$$

Prop. $\mathcal{O}^{\mathcal{B}}_X$ is a sheaf of rings on $\mathcal{B}$.

Pf: Uniqueness: use $V(\mathrm{Ann}(a)) = \{\mathfrak{p}\in \operatorname{Spec} R \mid a_{\mathfrak{p}} \neq 0 \text{ in } R_{\mathfrak{p}}\} = \operatorname{supp} a$

Gluing: 1) Wma $D(f) = X$.

2) Wma the indexing set to be finite, by qc-ness.

3) $X = \bigcup_{i=1}^m D(f_i)$, $t_i = \frac{a_i}{f_i^{n_i}} \in R_{f_i}$, wma $t_i = \frac{a_i}{f_i^n}$ for some common exponent n.

$\frac{a_i}{f_i^n} = \frac{a_j}{f_j^n} \in R_{f_if_j} \Rightarrow \exists M\geq 1: (f_if_j)^M\cdot(a_if_j^n - a_jf_i^n) = 0.$ in R

$$\underbrace{a_if_i^M}_{\tilde a_i} f_j^{n+M} - a_jf_j^Mf_i^{n+M} = 0 \quad \text{in } R$$

$\tilde a_i$, $\tilde n := M+n$

$\Rightarrow \frac{\tilde a_i}{f_i^{\tilde n}} = \frac{a_i}{f_i^n} = t_i$ but $\tilde a_j f_i^{\tilde n} - \tilde a_i f_j^{\tilde n} = 0$ in R

$\Rightarrow$ Wma $a_jf_i^n - a_if_j^n = 0$ in R.

4) $X = \bigcup_{i=1}^m D(f_i^n) \Rightarrow \sqrt{(f_1^n,\dots,f_m^n)} = R \Rightarrow \exists g_1,\dots,g_m: \; 1 = f_1^ng_1 + \dots + f_m^ng_m$

$t := \sum_{j=1}^m a_jg_j \in R$. We have $(t|_{D(f_i)})\cdot f_i^n = \sum_{j=1}^m f_i^n\cdot a_jg_j = \sum_{j=1}^m f_j^n a_ig_j = a_i\, 1$ in R_{f_i}

Def. The sheaf of rings on X induced by $\mathcal{O}^{\mathcal{B}}_X$ is denoted $\mathcal{O}_X$ and called the structure sheaf.

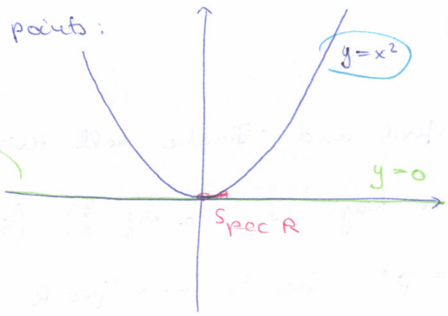
Concretely speaking: $\mathcal{O}_X(D(f)) = R_f$, and for $U\subseteq X$ general:

$$\mathcal{O}_X(U) = \left\{ s: U \longrightarrow \bigcup_{\mathfrak{p}\in U} R_{\mathfrak{p}} \;\middle|\; \begin{array}{l} s(\mathfrak{p})\in R_{\mathfrak{p}}\ \forall \mathfrak{p}\in U \text{ and } \exists \text{ cover } U = \bigcup_i D(f_i),\ t_i\in R_{f_i}, \\ s(\mathfrak{p}) = (t_i)_{\mathfrak{p}}\ \forall \mathfrak{p}\in D(f_i) \end{array} \right\}$$

Ex. A reg fn is not determined by its values at points:

$R = k[x]/(x^2)$

$k[x,y]/(y-x^2) \otimes_{k[x,y]} k[x] = R$ intersection (later)



$\operatorname{Spec} R = \{(x)\}$

$f = x\in R \Rightarrow f(p) = 0$ for the unique point $p = (x)\in \operatorname{Spec} R$ but $f\neq 0$.

More generally: $f\in R$, $f(p) = 0\ \forall \mathfrak{p}\in \operatorname{Spec} R \Rightarrow f\in \bigcap_{\mathfrak{p}\in\operatorname{Spec} R}\mathfrak{p} = \operatorname{nil}(R)$

Ex. $R = k[x]_{(x)}$, $\operatorname{Spec} R = \{(0), (x)\}$

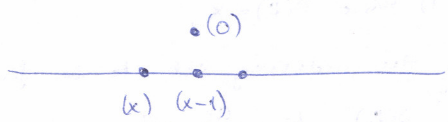
$$\mathcal{O}_X(D(x)) = \left(k[x]_{(x)}\right)_x = k(x)$$

$$\mathcal{O}_X(D(1)) = k[x]_{(x)}$$

Ex. $R = k[x]$, $\operatorname{Spec} R = \{(x-a)\} \cup \{(0)\}$, $D(f) = \{(x-a) \mid f(a) \neq 0\} \cup \{0\}$

$$\mathcal{O}_X(D(f)) = k[x]_f$$

$$\mathcal{O}_{X,\mathfrak{p}} = \begin{cases} k[x]_{(x-a)} & \mathfrak{p} = (x-a) \\ k[x]_{(0)} = k(x) & \mathfrak{p} = (0) \end{cases}$$



Schemes

22.11.2018

A map of underlying top spaces does not determine a pullback of functions.

Ex. $R = k[x_1, x_2]/(x_1^2, x_1x_2, x_2^2)$, $S = k[x]/(x^2)$

$\operatorname{Spec} R = \{(x_1^2, x_1x_2, x_2^2)\} = \{*\}$ $\quad\operatorname{Spec} S = \{(x)\}$

There is a unique map $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ but many different $R \longrightarrow S$:

$x_1 \mapsto ax$, $x_2 \mapsto bx$ determines such a map $\forall a, b \in k$

Def. A ringed space is a pair $(X, \mathcal{O}_X)$ where X is a top space, $\mathcal{O}_X$ a sheaf of rings on X. A morphism of rs is $(f, f^\#): (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ s.t. $f: X \longrightarrow Y$ is continuous and $f^\#: \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$ a morphism of sheaves of rings.

Def. Scheme: a rs $(X, \mathcal{O}_X)$ s.t. $\exists$ open cover $(U_i)_i$ of X for which $(U_i, \mathcal{O}_X|_{U_i}) \simeq (\operatorname{Spec} R_i, \mathcal{O}_{\operatorname{Spec} R_i})$ for some rings R_i.

A scheme is affine if the cover can be chosen to consist of one elt only.

Ex. $\coprod_{i \in I} \operatorname{Spec} R$

Warning. Mumford and Franke call this a prescheme.

Ex. $R \xrightarrow{\varphi} S$ in Ring. $\rightsquigarrow$ $(f_\varphi, f_\varphi^\#): (\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S}) \longrightarrow (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$

where $f_\varphi = \varphi^*: \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$
$\qquad\qquad\qquad\qquad \mathfrak{p} \longmapsto \varphi^{-1}(\mathfrak{p})$

and $f_\varphi^\#(U): \mathcal{O}_{\operatorname{Spec} R}(U) \longrightarrow \mathcal{O}_{\operatorname{Spec} S}(f_\varphi^{-1}(U))$ for any $U \subseteq \operatorname{Spec} R$

If $U = D(g)$, $f_\varphi^{-1}(D(g)) = D(\varphi(g))$,

$$\begin{array}{rcc} & \mathcal{O}_{\operatorname{Spec} R}(D(g)) = R_g & \frac{a}{g^n} \\ f_\varphi^\# := \varphi_g & \downarrow & \downarrow \\ & \mathcal{O}_{\operatorname{Spec} S}(D(\varphi(g))) = S_{\varphi(g)} & \frac{\varphi(a)}{\varphi(g)^n} \end{array}$$

In general:

let $s \in \mathcal{O}_{\operatorname{Spec} R}(U) = \left\{ s : U \longrightarrow \bigcup_{p \in U} R_p \;\middle|\; U = \bigcup_i D(g_i),\ t_i \in R_{g_i},\ s(p) = (t_i)_p \in R_{g_i} \right\}$

Need to define $f_\varphi^\#(U)(s) : f_\varphi^{-1}(U) \longrightarrow \bigcup_{q \in f_\varphi^{-1}(U)} S_q$ s.t.

if $U = \bigcup D(g_i)$ then $f_\varphi^{-1}(U) = \bigcup D(\varphi(g_i))$, $(f^\#(U)(s))(q) = \left(\varphi_{g_i}(t_i)\right)_q \in S_q$

This is well-def'd: if $q = f_\varphi(p)$ then consider $R_p \longrightarrow (R \setminus p)^{-1} S \longrightarrow S_q$, with composite φ_p.

Then
$$\begin{tikzcd} t_i \in R_{g_i} \arrow[r] \arrow[d, "\varphi_{g_i}"'] \arrow[dr, phantom, "G"] & R_p \arrow[d, "\varphi_p"] \\ S_{\varphi(g_i)} \arrow[r] & S_q \end{tikzcd}$$
because: $\varphi_{g_i}(t_i)_q = \varphi_p\left((t_i)_p\right) = \varphi_p(s(p))$ indep of i

Ex. Not every morph of rs is of this form:

$S = k(x)$, $\operatorname{Spec} S = \{*\}$

$R = k[x]_{(x)}$, $\operatorname{Spec} R = \{(0), (x)\}$

(+) $f : \operatorname{Spec} S \to \operatorname{Spec} R$, $* \mapsto (x)$

$f^\#(\{(0)\}) : \mathcal{O}_{\operatorname{Spec} R}(\{(0)\}) \longrightarrow \mathcal{O}_{\operatorname{Spec} S}(\emptyset) = 0$

$f^\#(\operatorname{Spec} R) : \underbrace{\mathcal{O}_{\operatorname{Spec} R}(\operatorname{Spec} R)}_{R} \longrightarrow k(x)$ the localisation

This f is not induced by any $\varphi : R \to S$. If it were, we would need to have $f((0)) = \varphi^{-1}((0)) = (x) \Rightarrow f_\varphi^\#(\operatorname{Spec} R) = \varphi$ is not injective since $(x) \in \operatorname{Ker}$.

Conclusion. Morphisms in RS are "too weak" for us.

Def. Locally ringed space: rs and $\forall p \in X$: $\mathcal{O}_{X,p}$ is a local ring.

Ex. $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is locally ringed: $\mathcal{O}_{\operatorname{Spec} R, p} = R_p$ which is of course local.

Ex. Every scheme is a lrs. (Because being a lrs is a local property.)

Def. A morphism of lrs is a morphism in RS s.t. $\forall p \in X$: the induced map $f_p^\#$:
$\mathcal{O}_{Y, f(p)} \longrightarrow \varinjlim_{p \in U} \mathcal{O}_X(f^{-1}(U)) \longrightarrow \mathcal{O}_{X,p}$ is a local ring homomorphism, i.e.
$\varphi_p^{-1}(m_p) = m_{f(p)}$.

Ex. For $R \xrightarrow{\varphi} S$ in Ring, the induced $(f_\varphi, f_\varphi^\#)$ is in LRS.

The map (+) is not in LRS.

Def. Morphism of schemes: morphism in LRS.

$U \subseteq X$, $g \in \mathcal{O}_X(U)$, $p \in U$. The val of g at p is:

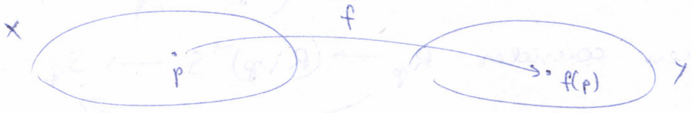
$$g(p) := \text{img of } g \text{ under } \mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,p} \longrightarrow \mathcal{O}_{X,p}/m_{X,p} = k(p) \quad \text{residue field}$$

$$V(g) := \{ p \in X \mid g(p) = 0 \}$$

<u>Lemma.</u> $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ LRS, $(f, f^\#): (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ morphism in RS.

Then $(f, f^\#)$ is in LRS $\iff \forall U \subseteq Y: \forall g \in \mathcal{O}_Y(U): f^{-1}(V(g)) = V(f^\#(g))$

Pf:



$$f^\#_p: \mathcal{O}_{Y, f(p)} \longrightarrow \mathcal{O}_{X,p}$$
$$m_{f(p)} \qquad\qquad m_p$$

Known: $(f^\#_p)^{-1}(m_p) \subseteq m_{f(p)}$. Nts $\supseteq$, or equivalently: $f^\#_p(m_{f(p)}) \subseteq m_p$.

Let $g \in m_{f(p)}$, $U \subseteq Y$ open, $f(p) \in U$, let $\tilde{g} \in \mathcal{O}_Y(U)$, $\tilde{g}_p = g$

$$g \in m_{f(p)} \Rightarrow f(p) \in V(\tilde{g}) \Rightarrow p \in f^{-1}(V(\tilde{g})) \overset{\text{ass.}}{\Rightarrow} p \in V(f^\#(\tilde{g}))$$
$$\Rightarrow f^\#(\tilde{g})_p = f^\#_p(g) \in m_p \Rightarrow f^\#(m_{f(p)}) \subseteq m_p$$
$$\Rightarrow p \in f^{-1}(V(g)) \iff f(p) \in V(g) \iff g_{f(p)} \in m_{f(p)} \underset{\text{ass.}}{=} (f^\#)^{-1}(m_p)$$
$$\iff f^\#(g_{f(p)}) \in m_p \iff p \in V(f^\#(g)).$$ □

<u>Thm.</u> X scheme, R ring. We have a bijection

$$\Theta: \mathrm{Hom}_{Sch}(X, \operatorname{Spec} R) \longrightarrow \mathrm{Hom}_{Ring}(R, \mathcal{O}_X(X))$$
$$(f, f^\#) \longmapsto \left(f^\#(\operatorname{Spec} R): \underset{\mathcal{O}_{\operatorname{Spec} R}(\operatorname{Spec} R)}{\overset{R}{\shortparallel}} \longrightarrow \mathcal{O}_X(X) \right)$$

<u>Cor.</u> $\mathrm{Hom}_{Sch}(\operatorname{Spec} S, \operatorname{Spec} R) = \mathrm{Hom}_{Ring}(R, S)$. In part, any morphism of LRS $\operatorname{Spec} S \to \operatorname{Spec} R$ is induced by a ring homomorphism $R \to S$.

Pf: Let $(f, f^\#): \operatorname{Spec} S \to \operatorname{Spec} R$ be a morphism in LRS, $\varphi := f^\#(\operatorname{Spec} R): R \to S$

Then $f^\#_\varphi(\operatorname{Spec} R) = \varphi = f^\#(\operatorname{Spec} R)$. By the injectivity of Θ we have $(f, f^\#) = (f_\varphi, f^\#_\varphi)$. □

PF OF THM:

<u>Injectivity:</u> Given $(f, f^\#): (X, \mathcal{O}_X) \longrightarrow (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$

Wts this is uniquely induced by $\varphi = f^\#(\operatorname{Spec} R): R \longrightarrow \mathcal{O}_X(X)$

<u>Step 1.</u> $x \in X$, $f(x) \in \operatorname{Spec} R$

$$\forall \mathfrak{p} \in \operatorname{Spec} R: \mathfrak{p} = \{ a \in R \mid a \in \mathfrak{p} \} = \{ a \in R \mid a(\mathfrak{p}) = 0 \}$$

$$\Rightarrow R \supseteq f(x) = \{a \in R \mid a(f(x)) = 0\} \overset{\text{lemma}}{\Rightarrow} (f^\# a)(x) = 0$$

$$\| \qquad \searrow a(f(x)) = 0 \Rightarrow f(x) \in V(a) \Rightarrow x \in V(f^\#(a))$$

$$\{a \in R \mid \varphi(a)(x) = 0\}$$

Step 2.: $f^\#$ is uq determined:

$$f^\#(U) : \mathcal{O}_{\operatorname{Spec} R}(U) \longrightarrow \mathcal{O}_X(f^{-1}(U))$$

If $U = D(g)$:

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\text{restr./loc}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{O}_X(X) \arrow[d, "\text{restr.}"] \\ R_g \arrow[r, "f^\#(D(g))"'] & \mathcal{O}_X(f^{-1}(D(g))) \end{tikzcd}$$

$\Rightarrow f^\#$ is unique by univ prop of localisation. – by determined by φ

In general: $U = \bigcup_{i \in I} D(g_i)$, $\operatorname{res}_{f^{-1}(D(g_i)), f^{-1}(U)} \circ f^\#(U)$ is uniquely determined by φ

$\Rightarrow$ so is $f^\#(U)$ by sheaf ax.

Surjectivity. Let $\varphi : R \longrightarrow \mathcal{O}_X(X)$ be given. Cover $X = \bigcup_i U_i$, $U_i \simeq \operatorname{Spec} S_i$.

Let $\varphi_i : R \longrightarrow \mathcal{O}_X(X) \xrightarrow{\text{res}} \mathcal{O}_X(U_i) = S_i$

Let $(f_i, f_i^\#) : U_i \longrightarrow \operatorname{Spec} R$ be the morphism induced by φ_i.

$$U_i \cong \operatorname{Spec} R_i \xrightarrow{(f_{\varphi_i}, f^\#_{\varphi_i})} \operatorname{Spec} R$$

Claim. These morphisms agree on overlaps $U_i \cap U_j$: $(f_i, f_i^\#)|_{U_i \cap U_j} = (f_j, f_j^\#)|_{U_i \cap U_j}$

Pf: $f_i^\#|_{U_i \cap U_j}(\operatorname{Spec} R) = \left(R \xrightarrow{\varphi} \mathcal{O}_X(X) \xrightarrow{\text{res}} \mathcal{O}_X(U_i) \longrightarrow \mathcal{O}_X(U_i \cap U_j)\right)$

$$= \left(R \xrightarrow{\varphi} \mathcal{O}_X(X) \xrightarrow{\text{res}} \mathcal{O}_X(U_j) \longrightarrow \mathcal{O}_X(U_i \cap U_j)\right) = f_j^\#|_{U_i \cap U_j}(\operatorname{Spec} R)$$

$$\begin{tikzcd} & \mathcal{O}_X(X) \arrow[dl] \arrow[dr] & \\ \mathcal{O}_X(U_i) \arrow[dr] \arrow[rr, phantom, "\circlearrowleft"] & & \mathcal{O}_X(U_j) \arrow[dl] \\ & \mathcal{O}_X(U_i \cap U_j) & \end{tikzcd}$$

By inj of $\odot$ we get the Claim.

By the Claim we can use the sheaf axiom to glue them together:

$f : X \longrightarrow \operatorname{Spec} R$ is def'd by $f|_{U_i} = f_i \quad \forall i \in I$.

Given $V \subseteq \operatorname{Spec} R$ and $s \in \mathcal{O}_{\operatorname{Spec} R}(V)$, define $f^\#(V)(s) := t \in \mathcal{O}_X(f^{-1}(V))$ s.t.

$$t|_{f^{-1}(V) \cap U_i} = f_i^\#(V)(s).$$

Cor. There is a contravar equivalence between the cat of affine schemes and the category of rings (commutative with 1).

Exc. $k = \bar{k}$. $\qquad$ PreVar $/k \xrightarrow{\ t\ }$ Sch $\quad$ fully faithful $\qquad$ 26.11.2018

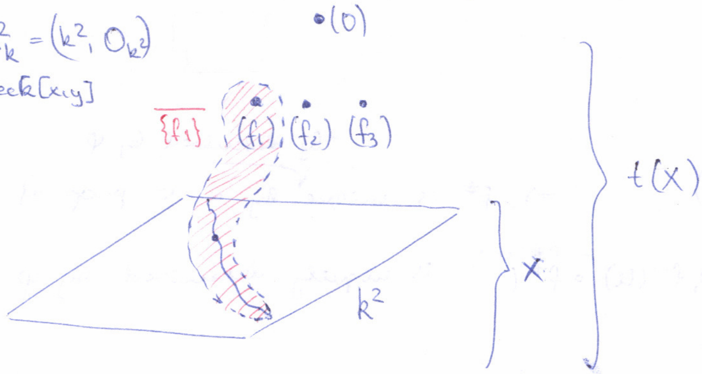
$$(X, \mathcal{O}_X) \longmapsto t(X)$$

$t(X) := \{[Y] \mid Y \subseteq X \text{ irred closed subset}\}$ "soberification"

Given $U \subseteq X$ open: $U^* := \{[Y] \mid Y \cap U \neq \emptyset\}$, $\mathcal{O}_{t(X)}(U^*) = \mathcal{O}_X(U)$

$\Rightarrow (t(X), \mathcal{O}_{t(X)})$ scheme and $\{p \in t(X) \mid k(p) = k\} = X$ in Set. (This was exc. 5/5(b))

Ex. $X = \mathbb{A}^2_k = (k^2, \mathcal{O}_{k^2})$
$= \operatorname{Spec} k[x,y]$



$\overline{\{f\}} = V(f) = \{\mathfrak{m} \subseteq k[x,y] \text{ max}, f \in \mathfrak{m}\} \cup \{(f)\}$

Def. S scheme. An <u>S-valued point</u> of a scheme X is a morphism $f : S \to X$.

$\underline{X(S)} := \{S\text{-val pts of } X\} = \operatorname{Hom}_{Sch}(S, X)$.

If $S = \operatorname{Spec} R$, we also call these <u>$R$-valued points</u> and write <u>$X(R)$</u>.

Ex. $X := \mathbb{A}^n = \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n]$ aff n-space

$S := \operatorname{Spec} K$, K field

$$\mathbb{A}^n(K) = \operatorname{Hom}_{Sch}(\operatorname{Spec} K, \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n])$$
$$\cong \operatorname{Hom}_{Ring}(\mathbb{Z}[x_1, \dots, x_n], K)$$
$$\cong K^n$$

More generally: R a ring, $\mathbb{A}^n(R) = R^n$

Even more generally: S any scheme, $\mathbb{A}^n(S) = \operatorname{Hom}_{Sch}(S, \mathbb{A}^n) = \operatorname{Hom}_{Ring}(\mathbb{Z}[x_1, \dots, x_n], \mathcal{O}_S(S)) = \mathcal{O}_S(S)^n$.

Ex. $X = \operatorname{Spec} \mathbb{Z}[x,y,z]/(x^m + y^m - z^m)$ $\quad (m \geq 1)$ $\quad X$: underlying geometric object of the eq. $x^m + y^m = z^m$

$X(R) = \operatorname{Hom}_{Ring}(\mathbb{Z}[x,y,z]/(x^m+y^m-z^m), R)$

$\cong \{(a,b,c) \in R^3 \mid a^m + b^m = c^m\}$

Note that we could have written anything instead of $\mathbb{Z}$, but this makes more sense due to $\operatorname{Spec} \mathbb{Z}$ being terminal.

More generally: $X(S) = \{(f_1, f_2, f_3) \in \mathcal{O}_S(S)^3 \mid f_1^m + f_2^m = f_3^m\}$ system of solutions parametrised by S

Rmk. Taking S-valued pts is functorial in S. More specifically, let $F : X \to Y$ be a morphism of schemes. Then

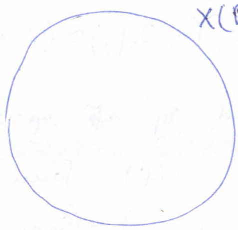
$$F_* : X(S) \longrightarrow Y(S)$$
$$(f : S \to X) \longmapsto (F \circ f : S \to Y)$$

defines a contravariant functor $Sch^{op} \to Set$.

<u>Ex.</u> $X = \mathrm{Spec}\left(\mathbb{Z}[x,y]/(x^2+y^2-1)\right)$ $\qquad$ $\mathbb{Z}[x,y]/(x^2+y^2-1)$

$$\begin{tikzcd} X \arrow[d, "F_\varphi"] \\ Y = \mathrm{Spec}(\mathbb{Z}[x,y]) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z}[x,y]/(x^2+y^2-1) \\ \mathbb{Z}[x,y] \arrow[u, "\varphi"] \end{tikzcd}$$

If $S = \mathrm{Spec}\,\mathbb{R}$ then $X(\mathbb{R}) \xhookrightarrow{F_\varphi} Y(\mathbb{R}) = \mathbb{A}^2(\mathbb{R}) = \mathbb{R}^2$ with img



$X(\mathbb{R})$

<u>Ex.</u> $X = \mathrm{Spec}\,\mathbb{Z}$, $X(S) = \mathrm{Hom}_{\mathrm{Sch}}(S, \mathrm{Spec}\,\mathbb{Z}) = \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}, \mathcal{O}_S(S)) = \{1 \mapsto 1\}$

$\Rightarrow$ $\mathrm{Spec}\,\mathbb{Z}$ is a terminal object in Sch

<u>Rmk.</u> X sch. $\rightsquigarrow$ $h_X : \mathrm{Sch}^{op} \longrightarrow \mathrm{Set}$

$$S \longmapsto \mathrm{Hom}_{\mathrm{Sch}}(S,X) = X(S)$$

We get $\mathrm{Sch} \longrightarrow \mathrm{Func}(\mathrm{Sch}^{op}, \mathrm{Set})$. Yoneda $\Rightarrow$ h is a fully faithful embedding (Exc. 6.5)

$$X \longmapsto h_X$$

$\Rightarrow$ Knowing all S-valued pts of X (and the maps b/w them) determines the scheme X.

In fact, $h' : \mathrm{Sch} \longrightarrow \mathrm{Func}(\mathrm{AffSch}^{op}, \mathrm{Set})$ is already fully faithful, so it suffices to

$X \longmapsto h_X|_{\mathrm{AffSch}}$ $\qquad$ know what X does on comm. rings.

<u>Fibre products</u>

<u>Def.</u> Let $\mathcal{C}$ be a category. Consider $\begin{tikzcd} & Y \arrow[d, "s"] \\ X \arrow[r, "r"] & S \end{tikzcd}$

A triple $(W \in \mathcal{C}, p_1 : W \longrightarrow X, p_2 : W \longrightarrow Y)$ is called a fibre product if $\begin{tikzcd} W \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "G"] & Y \arrow[d, "s"] \\ X \arrow[r, "r"'] & S \end{tikzcd}$

and satisfies the universal property

$$\begin{tikzcd} Z \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\forall q_2"] \arrow[ddr, bend right, "\forall q_1"'] & & \\ & W \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "s"] \\ & X \arrow[r, "r"'] & S \end{tikzcd}$$

<u>Ex.</u> $\mathcal{C} = \mathrm{Set}$. Then $X \times_S Y = \{(x,y) \in X \times Y \mid r(x) = s(y)\}$

<u>Ex.</u> $r : X \hookrightarrow S$, $s : Y \hookrightarrow S$ $\Rightarrow$ $X \times_S Y = X \cap Y \subseteq S$

<u>Ex.</u> $\{(x,y) \in X \times Y \mid r(x) = t,\ s(y) = t\} \simeq r^{-1}(t) \times s^{-1}(t) = X_t \times Y_t$ for $t \in S$

$$\begin{array}{l} \cap\!\! \mid \\ X \times_S Y \xrightarrow{r \circ p_1 = s \circ p_2} S \end{array}$$

<u>Fibre products for schemes</u>

$$(X \times_S Y)(T) = \mathrm{Hom}_{\mathrm{Sch}}(T, X \times_S Y) = \{(f,g) \in \mathrm{Hom}_{\mathrm{Sch}}(T,X) \times \mathrm{Hom}(T,Y) \mid \overbrace{r \circ f = s \circ g}^{r_* f = s_* g}\}$$

$$= \mathrm{Hom}_{\mathrm{Sch}}(T,X) \underset{\mathrm{Hom}_{\mathrm{Sch}}(T,S)}{\times} \mathrm{Hom}_{\mathrm{Sch}}(T,Y)$$

$$\begin{tikzcd} X \times_S Y \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "s"] \\ X \arrow[r, "r"'] & S \end{tikzcd}$$

Rmk. X sch, $U \subseteq X$ open $\Rightarrow (U, \mathcal{O}_X|_U)$ is a subscheme. Ntc: U is covered by affines.

Claim. $\underbrace{\text{Aff subsets}}$ form a basis of the topology.

$V \subseteq X$, $(V, \mathcal{O}_X|_V) \simeq (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$

1) X covered by aff opens
2) The opens $D(f)$ form a basis of $\operatorname{Spec} R$
3) $(D(f), \mathcal{O}_{\operatorname{Spec} R}|_{D(f)}) \simeq (\operatorname{Spec} R_f, \mathcal{O}_{\operatorname{Spec} R_f})$

$\Rightarrow$ Claim $\Rightarrow U$ is cov'd by affines. □

Thm. Sch has fibre products.

Pf: (O. admits this to be just a sketch)

1) Assume $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $S = \operatorname{Spec} C$. Then $X \times_S Y = \operatorname{Spec}(A \otimes_C B)$:

$$\operatorname{Hom}_{Sch}(T, \operatorname{Spec} A \otimes_C B) = \operatorname{Hom}_{Ring}(A \otimes_C B, \mathcal{O}_T(T)) = \operatorname{Hom}_{Ring}(A, \mathcal{O}_T(T)) \times_{\operatorname{Hom}(C, \mathcal{O}_T(T))} \operatorname{Hom}_{Ring}(B, \mathcal{O}_T(T)) =$$

$$= \operatorname{Hom}_{Sch}(T, \operatorname{Spec} A) \times_{\operatorname{Hom}_{Sch}(T, \operatorname{Spec} C)} \operatorname{Hom}_{Sch}(T, \operatorname{Spec} B) \quad \checkmark$$

this fibre product is taken in Set

2) In general, cover X, Y, S by affine schemes s.t. $r(X_i) \subseteq S_i$, $s(Y_i) \subseteq S_i$

$\Rightarrow X \times_S Y$ is obtained by gluing $X_i \times_{S_i} Y_i$. (Vast amount of technical details omitted.) □

If $X \times_S Y$ exists and we have

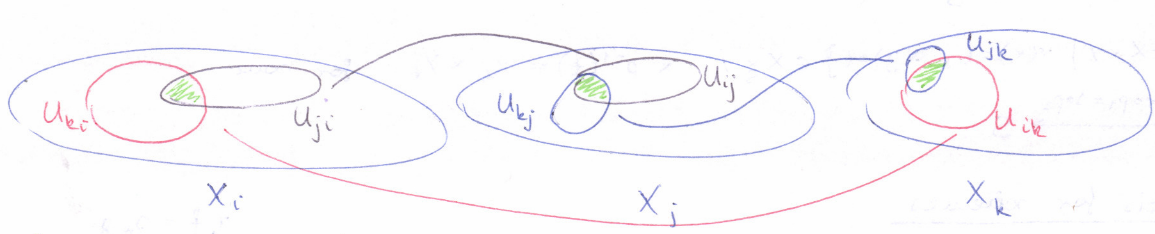
$$\begin{tikzcd} X \times_S Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d] & \supseteq Y_0 \text{ open} \arrow[dd] \\ X \arrow[r] & S & \\ U \text{ open } X \arrow[rr] & \text{open} & S_0 \end{tikzcd}$$

then

$$\begin{tikzcd} p_1^{-1}(X_0) \cap p_2^{-1}(Y_0) \arrow[r] \arrow[d] & Y_0 \arrow[d] \\ X_0 \arrow[r] & S_0 \end{tikzcd}$$

Gluing. If $\{X_i\}_{i \in I}$ is a faily of schemes, $U_{ji} \subseteq X_i$ opens $\forall j \neq i$, $\varphi_{ij}: U_{ij} \xrightarrow{\simeq} U_{ji}$ iso

s.t. $\varphi_{ij}^{-1} = \varphi_{ji}$, $\varphi_{jc}(U_{ji} \cap U_{ke}) = U_{ij} \cap U_{kj}$, $\varphi_{ki} = \varphi_{kj} \circ \varphi_{ji}$ on $U_{ji} \cap U_{ki}$



Then $\exists X$ scheme and $\psi_i: X_i \hookrightarrow X$ open embedding map,

$X = \bigcup_i \psi_i(X_i)$, $\psi_i(X_i) \cap \psi_j(X_j) = \psi_j(U_{ij})$, $\psi_j|_{U_{ij}} = \psi_i \circ \varphi_{ij}$

Pf: $X := \left(\coprod X_i\right) / (x \in U_{ij} \sim \varphi_{ij}(x) \in U_{ji})$ w/ quot top.

$W \subseteq X$ open $\Leftrightarrow \forall\, W \cap X_i$ open

$$\mathcal{O}_X(W) := \operatorname{Ker}\left(\prod_i \mathcal{O}_{X_i}(W \cap X_i) \longrightarrow \prod_i \mathcal{O}_{X_i}(W \cap U_{ji}) \right)$$

$$(s_i)_i \longmapsto \left(s_i|_{W \cap U_{ji}} - \varphi_{ji}^*(s_j|_{W \cap U_{ij}})\right)$$

$\mathcal{O}_X|_{X_i} = \mathcal{O}_{X_i}$

Ex. $\forall\, U_{ij} = \emptyset \Rightarrow X = \coprod X_i$, $\quad W \subseteq X$ open $\Leftrightarrow \forall\, W \cap X_i \subseteq X_i$ open

$$\mathcal{O}_X(W) = \prod_i \mathcal{O}_{X_i}(W \cap X_i)$$

Exc. $X_1 = \mathbb{A}^1$ with x, $\quad X_2 = \mathbb{A}^2$ with y

$U_1 = D(x) \qquad U_2 = D(y)$

$\varphi_{21} : U_1 \xrightarrow{\sim} U_2$

$y \longmapsto x \quad \rightsquigarrow$ line w/ 2 origins

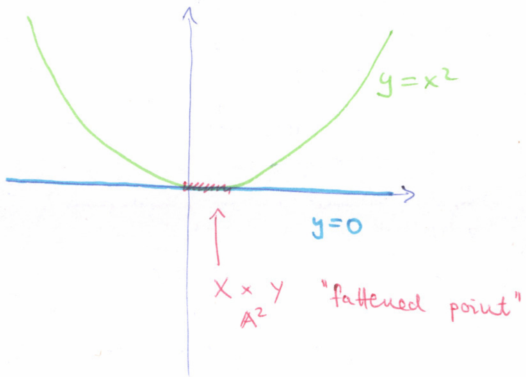
$\frac{1}{x} \longleftarrow x \quad \rightsquigarrow \mathbb{P}^1$

Ex. $X = \operatorname{Spec} \mathbb{Z}[x,y]/(y-x^2)$, $\quad Y = \operatorname{Spec} \mathbb{Z}[x,y]/(y=0)$

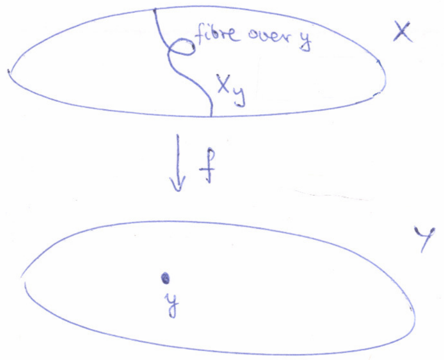
$$X \times_{\mathbb{A}^2} Y = \operatorname{Spec}\left(\mathbb{Z}[x,y]/(y-x^2, y)\right)$$

$$(X \times_{\mathbb{A}^2} Y)(k) = X(k) \times_{\mathbb{A}^1(k)} Y(k) = X(k) \cap Y(k)$$

Taking S-valued pts commutes with taking fibre products by the universal property.



Ex.



Giving $\operatorname{Spec} K \to Y$ is equivalent to giving $y \in Y$ and $k(y) \hookrightarrow K$. (Exc. sheet 5)

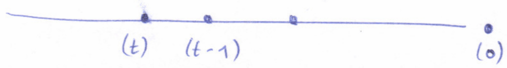
$$\begin{array}{ccc} X_y = X \times_Y \operatorname{Spec} k(y) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec} k(y) & \longrightarrow & Y \end{array}$$

In particular, take the following.

$X := \operatorname{Spec} k[x,y,t]/(ty-x^2)$

$\downarrow$

$\operatorname{Spec} k[t] = \mathbb{A}^1_k$



Fibre over $y = (t-\alpha)$: $\quad k(y) = k$

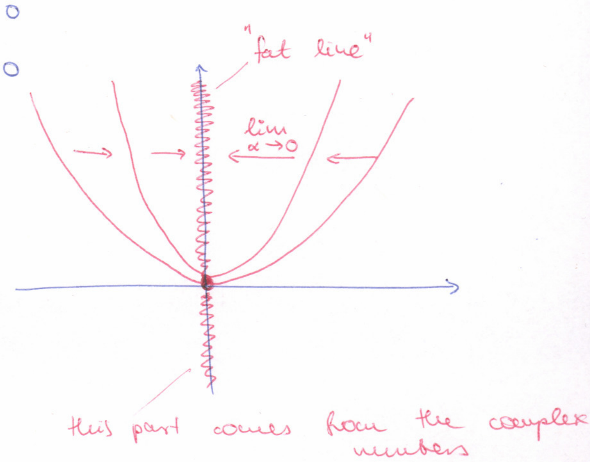
$$\Rightarrow X_y = \operatorname{Spec} k[x,y]/(\alpha y - x^2) = \begin{cases} \operatorname{Spec} k[x] = \mathbb{A}^1 & \alpha \neq 0 \\ \operatorname{Spec} k[x,y]/(x^2) & \alpha = 0 \end{cases}$$

$\{$Closed pts in $X_y\} = \{(x,y) \in k^2 \mid \alpha y - x^2 = 0\}$

Fibre of generic pt.: $\quad y = (0)$, $k(y) = \operatorname{Frac} k[t] = k(t)$

$$X_{(0)} = \operatorname{Spec}\left(k[x,y,t]/(yt-x^2) \otimes_{k[t]} k(t)\right)$$

$$= \operatorname{Spec}\left(k(t)[x,y]/(y = \tfrac{1}{t}x^2)\right) = \mathbb{A}^1_{k(t)}$$



29.11.2018

Def. A scheme over S is a morphism $f: X \longrightarrow S$.

How is X constrained by such a morphism $X \longrightarrow S$?

Claim. Giving $X \to \operatorname{Spec} R$ is equivalent to giving all rings $\mathcal{O}_X(U)$ (where $U \subseteq X$ open) the structure of R-algebra s.t. the restriction morphisms are R-algebra homomorphisms.

Pf: Given $X \xrightarrow{f} R$, define $R \longrightarrow \mathcal{O}_X(U)$ as the composition $R \xrightarrow{f^\#(\operatorname{Spec} R)} \mathcal{O}_X(X) \xrightarrow{res_{UX}} \mathcal{O}_X(U)$

By construction the restrictions are R-alg homs.

Conversely, given an R-alg structure on the $\mathcal{O}_X(U)$, we have $R \longrightarrow \mathcal{O}_X(X)$. Let $X \longrightarrow \operatorname{Spec} R$ be the associated morphism via $\operatorname{Hom}(X, \operatorname{Spec} R) \simeq \operatorname{Hom}(R, \mathcal{O}_X(X))$ □

Claim. Giving a morphism $f: X \longrightarrow Y$ over $\operatorname{Spec} R$, i.e.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "a"'] & \arrow[d, phantom, "\circlearrowleft"] & Y \arrow[dl, "a'"] \\ & \operatorname{Spec} R & \end{tikzcd} \quad (*)$$

is equivalent to giving a morphism $f: X \to Y$ s.t. $f^\#(U): \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ is a R-alg hom $\forall U \in \operatorname{Ouv}(X)$

Pf: Take global reg functions of $(*)$. Get: $a^\#(\operatorname{Spec} R) = f^\#(Y) \circ (a')^\#(\operatorname{Spec} R): R \to \mathcal{O}_X(X)$
$\Rightarrow f^\#(Y)$ is a R-alg hom.

In general:

$$\begin{tikzcd} \mathcal{O}_X(X) \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{O}_Y(Y) \arrow[l, "f^\#(Y)"'] \arrow[d, "res_{UY}"] \\ \mathcal{O}_X(f^{-1}(U)) \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{O}_Y(U) \arrow[l] \\ R \arrow[u, "(a|_{f^{-1}(U)})^\#"] \arrow[uu, bend left=60, "(a|_X)^\#"] \arrow[r, equal] & R \arrow[u, "(a'|_U)^\#"'] \arrow[uu, bend right=60, "(a'|_Y)^\#"'] \end{tikzcd}$$

The converse is similar. □

Def. S scheme. The cat Sch/S of schemes over S has objects $(X, X \longrightarrow S)$ where X is a scheme and $X \to S$ is a morphism of schemes, and morphisms

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & Y \arrow[dl] \\ & S & \end{tikzcd}$$

We write X/S for $X \to S$, also X/R for $X \to \operatorname{Spec} R$.

Ex. $\operatorname{Spec} \mathbb{Z}$ is final in Sch $\Rightarrow$ $Sch/\mathbb{Z} = Sch$.

Ex. There is no morphism $\operatorname{Spec} \mathbb{R} \to \operatorname{Spec} \mathbb{C}$. there is no ring hom $\mathbb{C} \to \mathbb{R}$ Hence one cannot make sense of $\operatorname{Spec} \mathbb{R}$ in $Sch/\mathbb{C}$, so in some sense, considering Sch/S instead of Sch constrains our world.

Def. T, X schemes $/S$. A T-valued pt of X is a cd

$$\begin{tikzcd} T \arrow[r] \arrow[dr] & X \arrow[d] \\ & S \end{tikzcd}$$

, i.e. a morph of S-schemes.

$X_S(T) := \{T\text{-valued pts of } X \text{ over } S\} = \operatorname{Hom}_{Sch/S}(T, X)$

Ex. $X = \operatorname{Spec} \mathbb{C}$

$$X(\mathbb{C}) = \operatorname{Hom}_{Sch}(\operatorname{Spec}\mathbb{C}, \operatorname{Spec}\mathbb{C}) \cong \underbrace{\operatorname{Hom}_{Ring}(\mathbb{C},\mathbb{C})}_{\text{very large}} \supseteq \operatorname{Gal}(\mathbb{C}/\mathbb{Q})$$

Consider $\operatorname{Spec}\mathbb{C}$ in $Sch/_{\operatorname{Spec}\mathbb{C}}$ via $\operatorname{Spec}\mathbb{C} \xrightarrow{id} \operatorname{Spec}\mathbb{C}$.

Then $X_{\mathbb{C}}(\mathbb{C}) = \{id\}$

Rmk. S is called a base scheme in this context. It may be dropped from the notation.

Ex. $k = \bar{k}$, X a prevar $/k$ $\Rightarrow$ $\mathcal{O}_X(U)$ is a k-alg. In particular, $t(X)$ is naturally a scheme over k. $t(X)_k(k) = \operatorname{Hom}_{Sch/k}(\operatorname{Spec} k, t(X)) = \{p \in t(X) \mid \underbrace{k(p) \hookrightarrow k}_{k\text{-alg hom.}}\} = \{p \in t(X) \mid k(p) = k\} = X$.

Ex. (Base change) $S' \longrightarrow S$ morphism, $X \in \operatorname{Ob} Sch/S$.

$$\begin{array}{ccc} S' \times_S X := X' & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \\ S' & \longrightarrow & S \end{array}$$

X' is obtained from X via base extension.

Properties of schemes

$|X| :=$ underlying top space

$X = (|X|, \mathcal{O}_X)$ scheme

Def. X is irreducible / connected / quasicompact if $|X|$ is.

Def. X is reduced if $\forall U \subseteq X$ open: $\mathcal{O}_X(U)$ is reduced (i.e. no nilpotents)

Equivalently, $\forall p \in X$: $\mathcal{O}_{X,p}$ is reduced.

Def. X is integral if $\forall U \subseteq X$ open: $\mathcal{O}_X(U)$ is an integral domain.

Exc. X integral $\Leftrightarrow$ reduced & irreducible.

Def. X is locally noetherian if $\exists (U_i)_i$ cover of aff open subschemes, $U_i = \operatorname{Spec} A_i$ s.t. $\forall A_i$ is noetherian.

Recall: if $U \subseteq X$ then $(U, \mathcal{O}_X|_U)$ is a scheme and called an open subscheme of X.

Def. X is noetherian if locally noetherian & qc ($\Rightarrow$ the cover can be made finite)

Prop. X loc noetherian $\Leftrightarrow$ $\forall \operatorname{Spec} A \subseteq X$ open affine subscheme A is noetherian

Key Lemma. Let P be a property of aff open subschemes of a scheme X for which

(1) if $A \subseteq X$ has P then $\operatorname{Spec} A_f$ has P

(2) if $A = \bigcup_i \operatorname{Spec} A_{f_i}$ s.t. $\forall \operatorname{Spec} A_{f_i}$ has P then $\operatorname{Spec} A$ has P.

Then if $X = \bigcup \operatorname{Spec} A_i$ s.t. $\forall \operatorname{Spec} A_i$ has P then every aff open subscheme of X has P.

We say that P is an affine local property.

Pf: Cf. AG I, Franke – Schwab – Wagner.

Pf OF PROP: Define P for Spec A to hold iff A is noetherian.

P is affine local:

(1) A noeth. $\Rightarrow$ A_f noeth. by correspondence of ideals

(2) Spec $A = \bigcup_i$ Spec A_{f_i}, $\forall A_{f_i}$ noeth. Then $(f_i \mid i \in I) = A \Rightarrow 1 = f_1 g_1 + \dots + f_\ell g_\ell$ for some $g_1, \dots, g_\ell \in A$. Let $I_1 \subset I_2 \subset \dots$ be a chain, $I := \bigcup_j I_j$

If $1 \in I$ then $1 \in I_{j_0}$ for some j_0, and we are done.

Otherwise $\exists k \in \{1, \dots, \ell\}$ s.t. $f_k \notin I$ $\Rightarrow$ get a chain $I_1' \subset I_2' \subset \dots$ in A_{f_k}, which must stabilise $\Rightarrow$ so does $I_1 \subset \dots$

The Prop. follows.

Relative properties of schemes

Ex. Finitely generatedness: $k(x)(y)$ fin gen over $k(x)$ but not over k

Def. $X \xrightarrow{f} Y$ locally of finite type if $\exists Y = \bigcup$ Spec B_i open affine cover and $f^{-1}(\text{Spec } B_i) \overset{(*)}{=} \bigcup_j (\text{Spec } A_{ij})$ open affine cover s.t. $\forall A_{ij}$ fin gen over B_i.

Moreover, f is of finite type if $(*)$ can be chosen to be finite.

Ex. Spec $\left(k[x_1, \dots, x_n] / f_1, \dots, f_n\right) \longrightarrow$ Spec k oft

$\coprod_{i \in I}$ Spec $k \longrightarrow$ Spec k loft but not oft when $\# I = \infty$

Spec $k[x_1, \dots]$ $\longrightarrow$ Spec k not loft

Prop. 1) oft $\Leftrightarrow$ loft + qc

2) f loft $\Leftrightarrow$ $\forall$ Spec $B \subseteq Y$ open $\forall$ Spec $A \subseteq f^{-1}(\text{Spec } B)$ open: A is a f.g. B-algebra.

Def. Quasicompact: preimage of affine open sets is quasicompact.

Pf OF PROP: 1) Omitted.

2) Say Spec B has P if $\exists$ affine open cover $f^{-1}(\text{Spec} B) = \bigcup$ Spec A_i s.t. $\forall A_i$ is f.g. $/B$.

Easy to check: P affine local. $\Rightarrow$ every open affine has P.

Let Spec $B \subseteq Y$ be open affine, and say Spec $A \subseteq f^{-1}(\text{Spec } B)$ has P' if A is a f.g. B-algebra.

Easy to check: P' is also affine local.